Supplementary Appendix (online only) to "Property Rights and Efficiency in OLG Models with Endogenous Fertility"

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The equilibrium (defined in Section 3.1.3 in the main text) can be defined in several different ways, in particular in sequence notation or as a recursive competitive equilibrium, and as a household problem, a dynastic problem or even a (modified) planner's problem. Since the household sequence problem is the easiest to interpret and because it is this formulation for which the efficiency concepts are defined, we stick to it throughout the paper. However, the dynastic problem or the planner's problem are the most convenient formulations for some of the technical proofs. To use these results in the paper, we need to formally establish that the various versions are equivalent. This is what we do in this Supplementary Appendix (see Figure S.1 for a graphical depiction). These results closely follow Alvarez (1994) but extend the setup to two periods of consumption and the presence of a minimum intergenerational transfer constraint.

More precisely, Section S.1 derives the equivalent of equation (4) from the point of view of the old household, which is convenient when specifying the boundedness condition for Assumption 1(e). Lemma S.1 in Section S.2 shows equivalence between the household and dynastic sequence problems. The latter formulation is used in Section S.3 to show that equation (4) uniquely defines a utility function U. To show that the first-order and transversality conditions for the equilibrium described in the main text are necessary and sufficient to characterize the equilibrium, we set up a Pseudo-Planner's problem in sequence and recursive form in Section S.4.1 and show that it is equivalent to the dynastic sequence problem. We then use standard dynamic programming techniques to characterize the Pseudo-Planner's value function and derive the necessary and sufficient conditions for optimality in Section S.4.2. Again using the Pseudo-Planner's problem, we show that decision rules and prices are continuous in the minimum transfer constraint, $\underline{b}$, in Section S.4.3. This result is needed for some of the proofs in Section 4.2 in the main text. For completeness, we also define a recursive competitive equilibrium (RCE, see Section S.5 Definition S.6). Proposition S.4 proves equivalence between the equilibrium defined in the main text and the RCE.

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S.1 Middle-aged and Old Household Problems

In this section, we show that the middle-aged household problem can be split into her problem when middle-aged and when old. In addition, we show that conditional on the transfer, an old agent agrees with her middle-aged children on how to split income between consumption, fertility, and savings. This is because altruism Ψ is specified as a function of children's utility, U_t. Therefore, rewriting the problem from the perspective of the old is equivalent to our original formulation.

Start with the problem of a middle-aged household in period t, as defined by the utility function (4) and the constraints (2) and (3). Since this is the household problem from the perspective of a middle-aged agent, we write the utility function here as $U_t^m = u(c_t^m) + \beta u(c_{t+1}^o) + \Psi(n_t, U_{t+1}^m)$. Define the utility when old as $U_t^o \equiv \beta u(c_t^o) + \Psi(n_{t-1}, U_t^m)$. Here, we can see that the old agent trades off c_t^m, c_{t+1}^o and n_t in the same way as her children. The only variable old parents and their middle-aged children in period t disagree on is c_t^o and, hence, the choice of b_t which determines how consumption is allocated inter-generationally. Given these insights, we can plug in for U_t^m, to get

$$U_t^o = \beta u(c_t^o) + \Psi(n_{t-1}, u(c_t^m) + \beta u(c_{t+1}^o) + \Psi(n_t, U_{t+1}^m))$$

and use the definition of U_{t+1}^o to derive

$$U_t^o = \beta u(c_t^o) + \Psi(n_{t-1}, u(c_t^m) + U_{t+1}^o). \tag{S.1}$$

This logic shows that the two formulations are equivalent, which turns out to be useful for some of the proofs.

S.1.1 Old Household Sequence Problem

In this subsection we specify the old household's sequence problem. Since $u(.)$ is strictly increasing, the budget constraints will hold with equality. Given this, let us define the optimal transfer function, $\widetilde{b}$,

$$\widetilde{b}(z, x_t, x_{t+1}; w_t, r_t) \equiv \underset{b_t \geq \underline{b}}{argmax}\{\beta u(r_t s_t - n_{t-1} b_t w_t) + \Psi(n_{t-1}, u(w_t(1+b_t) - s_{t+1} - \theta_t n_t) + z). \tag{S.2}$$

where $x_t = (s_t, n_{t-1})$. Recall our notation for sequences of individual states, $\underset{\sim}{x}_t = \{x_k\}_{k=t}^{\infty}$. Let the set of budget feasible allocations from $x_t = (s_t, n_{t-1})$ be given by

$$\Pi(x_t, \underset{\sim}{w}_t, \underset{\sim}{r}_t) = \{\underset{\sim}{x}_t : (s_{k+1} + \theta_k n_k)n_{k-1} \leq w_k n_{k-1} + r_k s_k, \forall k \geq t, x_t \text{ given}\}.$$

Hence, the utility of an old agent in equation (S.1) can be written as a function of the infinite sequence of state variables $\underset{\sim}{x}_t \in \Pi(x_t, \underset{\sim}{w}_t, \underset{\sim}{r}_t)$, and prices $(\underset{\sim}{w}_t, \underset{\sim}{r}_t)$:

$$\begin{aligned} U^o(\underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t) &= \beta u(r_t s_t - n_{t-1}\widetilde{b}(.)w_t) \\ &\quad + \Psi(n_{t-1}, u(w_t(1 + \widetilde{b}(.)) - s_{t+1} - \theta_t n_t) + U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1})) \end{aligned} \tag{S.3}$$

where $\underset{\sim}{x}_t = (x_t, \underset{\sim}{x}_{t+1})$, i.e. $\underset{\sim}{x}_{t+1}$ agrees with $\underset{\sim}{x}_t$ from $t+1$ on. If we can show that equation (S.3) defines a unique function $U^o(.)$ for all $\underset{\sim}{x}_t \in \Pi(x_t, \underset{\sim}{w}_t, \underset{\sim}{r}_t)$, then we can deduce the unique function

$$U^m(b_t, \underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t) = u(w_t(1 + b_t) - s_{t+1} - \theta_t n_t) + U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1}) \tag{S.4}$$

for all $\underset{\sim}{x}_t \in \Pi(x_t, \underset{\sim}{w}_t, \underset{\sim}{r}_t)$ and $b_t = \widetilde{b}(.)$. Note that U^m is only defined for rationalizable transfers $\widetilde{b}$, as is standard in models with recursively defined preferences. We deal with uniqueness in Section S.3.

We can formally state the boundedness condition in Assumption 1(e). This is a natural extension of the condition in Assumption 4.11. in Stokey, Lucas, and Prescott (1989) (Chapter 4, Section 3), Assumption 1d in Alvarez and Stokey (1998) or Assumption 5′ in Alvarez (1994). Let $||.||_E$ denote the Euclidean norm and $|.|$ the absolute value.

Assumption S.1 *The objective is bounded in the sense that* $\exists B \in (0, \infty)$ *such that for all* $t \geq 0$, *for all* $x_{t+1} = (s_{t+1}, n_{t-1})$ *s.t.* $(s_{t+1} + \theta_t n_t)n_{t-1} \leq w_t n_{t-1} + r_t s_t$ *and for all* $z \in \mathbb{R}$ *we have*

$$\begin{aligned} &|\beta u(r_t s_t - n_{t-1}\widetilde{b}(.)w_t) + \Psi(n_{t-1}, u(w_t(1 + \widetilde{b}(.)) - s_{t+1} - \theta_t n_t) + z)| \\ &\qquad \leq B\{(||(s_t, n_{t-1}, 1)||_E)^\nu + n_{t-1}^\nu(||(s_{t+1}, n_t, 1)||_E)^\nu + |z|\} \end{aligned}$$

where $\widetilde{b}(.)$ *as defined in (S.2).*

Finally, we define the old household's sequence problem in Definition S.1 where the left-hand side is the optimized value[44] as a function of the initial conditions s_0 and n_0, while the right-hand side describes the choice problem, where the objective function is maximized over the entire sequences $\underset{\sim}{s}_0$ and $\underset{\sim}{n}_0$.

[44] We use U^{o**} here because U^{o*} is used for the optimized value of the recursive problem.

Definition S.1 *Old Household Sequence Problem (HSP):*

$U^{o**}(s_0, n_0; \underset{\sim}{w}_0, \underset{\sim}{r}_0)) = \sup\{U^o(\underset{\sim}{s}_0, \underset{\sim}{n}_0; \underset{\sim}{w}_0, \underset{\sim}{r}_0) : (\underset{\sim}{s}_0, \underset{\sim}{n}_0) \in \Pi(s_0, n_0; \underset{\sim}{w}_0, \underset{\sim}{r}_0))\}$

S.2 Household and Dynastic Sequence Problems

In this section, we show that the old household problem in sequence form (described in Section S.1.1) can be written in dynasty aggregates. Note that this results in a convex constrained set which is useful because it allows us to prove that U_D^o is uniquely determined over budget feasible sequences and implies a unique household value function U that solves equation (4). We do this in Section S.3. This formulation is also the basis from which the Pseudo-Planner's problem (in sequence and recursive form) is derived. The latter allows us to use standard techniques to characterize the equilibrium allocation, which we do in Sections S.4.2 and S.4.4.

Let us define the primitives of the dynastic problem. Normalize the measure of initial old to $N_0^o = 1$. Let N_t^m and N_t^o be the number of middle age and old descendants of the initial old in period t, respectively: $N_t^m = N_{t-1}^m n_{t-1}$ and $N_{t-1}^m = N_t^o$. Let $C_t^m \equiv c_t^m N_t^m$ and $C_t^o \equiv c_t^o N_t^o$ be the total consumption of the middle- and old-aged descendants in period t, respectively. Also define $S_t \equiv N_t^o s_t$ and $B_t \equiv N_t^m b_t$ as the dynastic savings and transfers. The dynastic state is $x_{Dt} \equiv (S_t, N_t^m, N_t^o)$. Note that this relates to the individual state as follows: $x_{Dt} = (N_t^o s_t, N_t^o n_{t-1}, N_t^o) = (N_t^o x_t, N_t^o) = N_t^o(x_t, 1)$. Analog to the household problem, we let $\underset{\sim}{x}_{Dt}$ denote the sequence of dynastic states.

Recall that by Assumption 1, Ψ is either homogenous of degree ν or separable in its two arguments. For the h.o.d. ν case (i), define the period utility, altruism, and dynastic utility functions as follows.

$$\begin{aligned} u_D(C, N) &\equiv (N)^\nu u\left(\frac{C}{N}\right) \\ \Psi_D(N, U) &\equiv \Psi\left(N, \frac{U}{N^\nu}\right) \\ U_D^o(\underset{\sim}{x}_{Dt}; \underset{\sim}{w}_t, \underset{\sim}{r}_t) &\equiv (N_t^o)^\nu U_t^o(\underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t). \end{aligned} \tag{S.5}$$

Then it is easy to show that Ψ being homogeneous of degree ν in n (i.e. Assumption 1(c)(i)) implies that Ψ_D is homogeneous of degree ν in the sense that $\Psi_D(\lambda N, \lambda^\nu U) = \lambda^\nu \Psi_D(N, U)$ for all $\lambda > 0$ and $\nu \neq 0$.

We now deal with the non-separable case (ii) from Assumption 1(c). Note that for several of the proofs we need Ψ_D to be homogenous of degree ν, which with the definition of Ψ_D would not be true for case (ii). However, with some appropriate redefinitions, case (ii) can also be written as a homogeneous function. Here (and in Section S.2.1) we briefly discuss how this works without going into too much detail to avoid tedious notation and repetition. Define two new functions that depend only on n and U respectively: $h(n)$ and $\widetilde{\Psi}(U)$ so that $\Psi(n,U) = h(n) + \widetilde{\Psi}(U)$. Since by construction $\widetilde{\Psi}$ does not depend on n, we can also write $\widetilde{\widetilde{\Psi}}(n,U)$ and assume that $\widetilde{\widetilde{\Psi}}$ is h.o.d. of degree $\nu = 0$ in n. This way of writing it assures that we can use the same steps in the proofs as for the non-separable case where $\Psi(n,U)$ is assumed to be h.o.d. of degree ν in n. Specifically, for the separable case (ii), define the dynastic functions as follows:

$$\Psi_D(N,U) \equiv \widetilde{\widetilde{\Psi}}\left(N, \frac{U}{N^\nu}\right).$$

Further, define

$$h_D(N^o, N^m) \equiv h\left(\frac{N^m}{N^o}\right).$$

Clearly, h_D is homogeneous of degree 0 in (N^o, N^m). It is also easy to show that $\widetilde{\widetilde{\Psi}}(n,U)$ being h.o.d. $\nu = 0$ in n implies that Ψ_D has the same property as in case (i) but with $\nu = 0$. These are the properties needed for several of the proofs.

Then, it is straightforward to show that Assumption 1 is equivalent to:

Assumption S.2

(a) $u_D(.,.)$ *is continuous, strictly concave, continuously differentiable, increasing, and* $u'_D(0,N) = \infty$.

(b) $\Psi_D(N,U)$ *is continuous, strictly concave in* n *and weakly concave in* U, *continuously differentiable, and strictly increasing in both arguments.*

(c) Ψ_D *is homogeneous of degree* ν *in the sense that*
$\Psi_D(\lambda N, \lambda^\nu U) = \lambda^\nu \Psi_D(N,U), \forall \lambda > 0, N \in \mathbb{R}_+, U \in \mathbb{R}$
with either (i) or (ii):

  (i) $\nu < 1, \nu \neq 0$ *and* u_D *is strictly increasing in* N;

  (ii) $\nu = 0$ *and* $h_D(N^o, N^m)$ *is homogeneous of degree 0.*

(d) $\Psi_D(N,U)$ *discounts at rate* $\zeta < 1$ *in the sense that*

$\forall N \in \mathbb{R}_+, U \in \mathbb{R}, a > 0, \Psi_D(N, U + a) \leq \Psi_D(N,U) + \zeta a.$

(e) The objective satisfies a boundedness condition on the set of budget feasible allocations.

S.2.1 Dynastic Utility

Start with the utility function of an old household in period t defined in equation (S.1) and multiply with $(N_t^o)^\nu$ to get

$$(N_t^o)^\nu U_t^o = (N_t^o)^\nu \beta u(c_t^o) + (N_t^o)^\nu \Psi(n_{t-1}, u(c_t^m) + U_{t+1}^o).$$

From the definition of Ψ_D, we have $\Psi(n_{t-1}, u(c_t^m) + U_{t+1}^o)) = \Psi_D(n_{t-1}, n_{t-1}^\nu u(c_t^m) + n_{t-1}^\nu U_{t+1}^o)$. Thus,$(N_t^o)^\nu U_t^o = (N_t^o)^\nu \beta u(c_t^o) + (N_t^o)^\nu \Psi_D(n_{t-1}, n_{t-1}^\nu u(c_t^m) + n_{t-1}^\nu U_{t+1}^o)$.

For the non-separable case (i), using the homogeneity in Assumption S.2.(c), we can rewrite the problem as

$$(N_t^o)^\nu U_t^o = \beta(N_t^o)^\nu u(c_t^o) + \Psi_D(N_t^o n_{t-1}, (N_t^o)^\nu n_{t-1}^\nu u(c_t^m) + (N_t^o)^\nu n_{t-1}^\nu U_{t+1}^o).$$

Using the definitions of C_t^m, C_t^o, N_t^m, and N_t^o, rewrite as

$$(N_t^o)^\nu U_t^o = \beta(N_t^o)^\nu u\left(\frac{C_t^o}{N_t^o}\right) + \Psi_D(N_t^m, (N_t^m)^\nu u\left(\frac{C_t^m}{N_t^m}\right) + (N_{t+1}^o)^\nu U_{t+1}^o).$$

From the definitions of u_D and U_{Dt}^o, this is

$$U_{Dt}^o = \beta u_D(C_t^o, N_t^o) + \Psi_D(N_t^m, u_D(C_t^m, N_t^m) + U_{Dt+1}^o). \tag{S.6}$$

For the separable case (ii), the expression needs to be slightly modified

$$U_{Dt}^o = \beta u_D(C_t^o, N_t^o) + h_D(N_t^o, N_t^m) + \Psi_D(N_t^m, u_D(C_t^m, N_t^m) + U_{Dt+1}^o). \tag{S.7}$$

In the remaining proofs we focus on the non-separable case to avoid having to carry $h_D(N_t^o, N_t^m)$ around. However, the logic works exactly the same with this additional term.

S.2.2 Dynastic Constraints

Consider the constraint set from the household problem:

$$\begin{aligned}
&c_t^m + \theta_t n_t + s_{t+1} \le w_t(1+b_t) \\
&c_{t+1}^o + n_t b_{t+1} w_{t+1} \le r_{t+1} s_{t+1} \\
&b_{t+1} \ge \underline{b}_{t+1} \\
&c_t^m, c_{t+1}^o, n_t \ge 0.
\end{aligned}$$

Use the definition of aggregate variables and multiplying through, this is

$$\begin{aligned}
&C_t^m + \theta_t N_{t+1}^m + S_{t+1} \le w_t \left(N_t^m + B_t\right) \\
&C_{t+1}^o + B_{t+1} w_{t+1} \le r_{t+1} S_{t+1} \\
&B_{t+1} \ge \underline{b}_{t+1} N_{t+1}^m \\
&C_t^m, C_{t+1}^o, N_{t+1}^m, N_t^m \ge 0.
\end{aligned}$$

Using these transformations, we can now specify an old dynastic sequence problem.

S.2.3 Dynastic and Old Household Sequence Problems: Equivalence

Since $u_D(.)$ is strictly increasing, the budget constraints will hold with equality. Given this, let us define the optimal transfer function, $\widetilde{B}$, $\forall t \ge 0, z \in \mathbb{R}$,

$$\widetilde{B}(z, x_{Dt}, x_{Dt+1}; w_t, r_t) \equiv \underset{B_t \ge \underline{b}_t N_t^m}{argmax} \{\beta u_D(r_t S_t - B_t w_t, N_t^o) + \Psi_D(N_t^m, u_D(w_t(N_t^m + B_t) - S_{t+1} - \theta_t N_{t+1}^m, N_t^m) + z) \tag{S.8}$$

where $x_{Dt} = (S_t, N_t^m, N_t^o)$. Note that given Assumption S.2(c), we have

$$\widetilde{B}(\lambda^\nu z, \lambda x_{Dt}, \lambda x_{Dt+1}; w_t, r_t) = \lambda \widetilde{B}(z, x_{Dt}, x_{Dt+1}; w_t, r_t). \tag{S.9}$$

Note that this homogeneity property is specific to the dynastic problem and will help in the proofs. In particular, it will be used to show that the recursively defined utility is homogeneous of degree ν.

Recall our notation that $\underset{\sim}{x}_{Dt} = \{x_{Dk}\}_{k=t}^{\infty}$. Let the set of budget feasible allocations from $x_{Dt} = (S_t, N_t^m, N_t^o)$ be given by

$$\Pi_D(x_{Dt}, \underset{\sim}{w}_t, \underset{\sim}{r}_t) \equiv \{\underset{\sim}{x}_{Dt} : S_{k+1} + \theta_k N_{k+1}^m \le w_k N_k^m + r_k S_k, \forall k \ge t, x_{Dt} \text{ given}\}.$$

From the derivations in the previous subsection and given our definitions of aggregate variables, we have $\underset{\sim}{x}_t \in \Pi_D(x_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t)$ if and only if $\underset{\sim}{x}_{Dt} \in \Pi_D(x_{Dt}; \underset{\sim}{w}_t, \underset{\sim}{r}_t)$. Also, $b_t \geq \underline{b}_t$ if and only if $B \geq N_t^m \underline{b}_t$. Hence, the dynastic utility as a function of the infinite sequence of state variables $\underset{\sim}{x}_{Dt} \in \Pi_D(x_{Dt}, \underset{\sim}{w}_t, \underset{\sim}{r}_t)$, and prices $(\underset{\sim}{w}_t, \underset{\sim}{r}_t)$, can be written as

$$\begin{aligned} U_D^o(\underset{\sim}{x}_{Dt}; \underset{\sim}{w}_t, \underset{\sim}{r}_t) \;&=\; \beta u_D(r_t S_t - \widetilde{B}(.) w_t, N_t^o) \\ &\quad + \Psi_D(N_t^m, u_D(w_t(N_t^m + \widetilde{B}(.)) - S_{t+1} - \theta_t N_{t+1}^m, N_t^m) + U_D^o(\underset{\sim}{x}_{Dt+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1})) \end{aligned} \tag{S.10}$$

where $\underset{\sim}{x}_{Dt} = (x_{Dt}, \underset{\sim}{x}_{Dt+1})$, i.e. $\underset{\sim}{x}_{Dt+1}$ agrees with $\underset{\sim}{x}_{Dt}$ from $t+1$ on.

Note that, if we can show that equation (S.10) defines a unique function $U_D^o(.)$ for all $\underset{\sim}{x}_{Dt} \in \Pi(x_{Dt}, \underset{\sim}{w}_t, \underset{\sim}{r}_t)$ (which we do in Section S.3), then we can deduce the unique function

$$U^o(\underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t) = \frac{U_D^o(\underset{\sim}{x}_{Dt}; \underset{\sim}{w}_t, \underset{\sim}{r}_t)}{(N_t^o)^\nu} \tag{S.11}$$

that solves equation (S.3), where $\underset{\sim}{x}_t$ is related to $\underset{\sim}{x}_{Dt}$ as explained above.

We are now ready to formally state the boundedness condition in Assumption S.2(e). This is a natural extension of the condition in Assumption 4.11. in Stokey, Lucas, and Prescott (1989) (Chapter 4, Section 3), Assumption 1d in Alvarez and Stokey (1998) or Assumption 5' in Alvarez (1994). Let $||.||_E$ denote the Euclidean norm.

Assumption S.3 *The objective is bounded in the sense that* $\exists B \in (0, \infty)$ *such that for all* $t \geq 0$*, for all* (x_{Dt}, x_{Dt+1}) *such that* $[S_{t+1} + \theta_t N_{t+1}^m \leq w_t N_t^m + r_t S_t]$ *and for all* $z \in \mathbb{R}$ *we have*

$$\begin{aligned} &|\beta u_D(r_t S_t - \widetilde{B}(.) w_t, N_t^o) + \Psi_D(N_t^m, u_D(w_t(N_t^m + \widetilde{B}(.)) - S_{t+1} - \theta_t N_{t+1}^m, N_t^m) + z)| \\ &\qquad \leq B\{(||x_{Dt}||_E)^\nu + (||x_{Dt+1}||_E)^\nu + |z|\} \end{aligned}$$

where $\widetilde{B}(.)$ *as defined in (S.8).*

Next, we define the dynastic sequence problem as, where as in Definition S.1 the right-hand side is the choice problem and U_D^{o**} denotes the optimized value.

Definition S.2 *Dynastic Sequence Problem (DSP):*

$U_D^{o**}(x_{D0}; \underset{\sim}{w}_0, \underset{\sim}{r}_0)) = \sup\{U_D^o(\underset{\sim}{x}_{D0}; \underset{\sim}{w}_0, \underset{\sim}{r}_0) : \underset{\sim}{x}_{D0} \in \Pi_D(x_{D0}; \underset{\sim}{w}_0, \underset{\sim}{r}_0))\}$.

Lemma S.1 *The problems defined in Definitions S.1 and S.2 are equivalent.*

Proof. Let $\underset{\sim}{x}_0 \in \Pi(x_0, \underset{\sim}{w}_0, \underset{\sim}{r}_0)$ and $\underset{\sim}{x}_{D0} \in \Pi_D(x_{D0}, \underset{\sim}{w}_0, \underset{\sim}{r}_0)$ be any budget feasible sequences such that $s_t = S_t/N_t^o$ and $n_{t-1} = N_t^m/N_t^o$. Then by definition we have $U_D^o(x_{D0}, \underset{\sim}{w}_0, \underset{\sim}{r}_0) = (N_0^o)^\nu U^o(x_0, \underset{\sim}{w}_0, \underset{\sim}{r}_0)$. Since N_0^o is part of the initial condition, their suprema, $\underset{\sim}{x}_0^*$ and $\underset{\sim}{x}_{D0}^*$

are also such that $s_t^* = S_t^*/N_t^{o*}$ and $n_{t-1}^* = N_t^{m*}/N_t^{o*}$. See also Remark 1 p.20 in Alvarez (1994). ■

Next, to state Assumption 2 more precisely we need some preliminaries. Define κ, the minimum or maximum rate of change of x_D as follows.

Definition S.3 $\overline{\kappa} \equiv \sup\{||x_{Dt+1}||_E/||x_{Dt}||_E, (x_{Dt}, x_{Dt+1})$ *s.t.* $[S_{t+1} + \theta_t N_{t+1}^m \leq w_t N_t^m + r_t S_t]\}$

$\underline{\kappa} \equiv \inf\{||x_{Dt+1}||_E/||x_{Dt}||_E, (x_{Dt}, x_{Dt+1})$ *s.t.* $[S_{t+1} + \theta_t N_{t+1}^m \leq w_t N_t^m + r_t S_t]\}$

If $\nu > 0$*, define* $\kappa \equiv \overline{\kappa}$*, if* $\nu < 0$*, define* $\kappa \equiv \underline{\kappa}$*.*

We are now ready to state the boundedness condition in Assumption 2 formally.

Assumption S.4 *Assume* $\kappa > 0$ *and* $\kappa^\nu \zeta < 1$*.*

The intuition of this bound is as follows. If $\nu > 0, U_D^o \geq 0$, and we have to make sure not to go to $+\infty$ by bounding the growth rate of the state variable. If, on the other hand, $\nu < 0, U_D^o \leq 0$, and we have to make sure not to go to $-\infty$ by bounding the drop rate of the state variable. It is easy to see that Definition S.3 implies the following remark:

Remark 1 $\forall \nu < 1, \kappa^\nu \geq (||x_{Dt+1}||_E/||x_{Dt}||_E)^\nu$.

Remark 1 together with Assumptions S.3 and S.4 will be used in several proofs showing the utility is bounded.

S.2.4 Dynastic Formulations for BB and RB

For the utility specifications introduced in Section 3.3, the dynastic formulation can be expressed as follows. Assuming $u(x) = \frac{x^{1-\sigma}}{1-\sigma}$, $g(x) = x^{1-\varepsilon}$ and $h(x) = u(x)$, we get:

$$BB \qquad U_{D,s}^{oBB} = \sum_{t=s}^{\infty} \zeta^{t-s} \left[\frac{\beta (N_t^o)^{\sigma-\varepsilon} (C_t^o)^{1-\sigma} + \zeta (N_t^m)^{\sigma-\varepsilon} (C_t^m)^{1-\sigma}}{1-\sigma} \right] \tag{S.12}$$

$$RB \qquad U_{D,s}^{oRB} = \sum_{t=s}^{\infty} \zeta^{t-s} \left[\frac{(N_t^o)^{\sigma-1} [\beta (C_t^o)^{1-\sigma} + \gamma (N_t^m)^{1-\sigma}] + \zeta (N_t^m)^{\sigma-1} (C_t^m)^{1-\sigma}}{1-\sigma} \right]$$

which are the utility functions corresponding to equation (S.10).

By assumptions S.2(a) and (b) we need that the utility is strictly increasing and strictly concave in all its arguments, $\{N_t^m, C_t^m, C_t^o\}_{t=s}^\infty$. Some of these conditions are useful in comparing the two specifications. In particular, for utility to be strictly increasing in N_t^m in the $RB-$altruism, the condition boils down to:

$$RB. \qquad \gamma u'(n_t) > \zeta \frac{\left[u'(c_{t+1}^m)c_{t+1}^m + \beta u'(c_{t+2}^o)c_{t+2}^o + \gamma u'(n_{t+1})n_{t+1}\right]}{n_t}. \tag{S.13}$$

This condition says that the direct utility benefit has to be strictly larger than the indirect utility cost of diluting per generation consumption and fertility one period later.

With logarithmic utility, this condition boils down to:

$$RB(log). \qquad \gamma > \tfrac{\zeta(1+\beta)}{1-\zeta}. \tag{S.14}$$

Following Jones and Schoonbroodt (2010), there are three sets of joint parameter restrictions that ensure that utility satisfies the desired monotonicity and concavity properties for $BB-$type altruism:

$$\begin{aligned} &BB.1 \quad 0 < \varepsilon < \sigma < 1; \\ &BB.2 \quad 1 < \sigma < \varepsilon; \\ &BB.3 \quad 1 - \varepsilon = \delta(1-\sigma) \text{ for some } \delta > 1 \text{ and } \sigma \to 1. \end{aligned} \tag{S.15}$$

In the last case, utility is separable and logarithmic and hence equivalent to the RB specification with logarithmic utility with $\gamma \equiv \frac{\delta\zeta(1+\beta)}{1-\zeta}$. Since $\delta > 1$, condition (S.14) is satisfied.[45]

Further, it is straightforward to verify that utility is homogeneous of degree $\nu = 1-\varepsilon$ in the BB—case and homogeneous of degree $\nu = 0$ in the RB—case. Hence, Asumption S.2(c) is satisfied. Finally, Assumption S.2(d) is satisfied if

$$\zeta < 1. \tag{S.16}$$

S.3 Equation (4) uniquely defines a utility function U

The next proposition states that the function U that we implicitly defined by equation (4) exists and is unique. Some definitions are needed. Let $\mathbb{X}$ be the set of budget feasible sequences, $\underset{\sim}{x}_0 \in \Pi(x_0, \underset{\sim}{w}_0, \underset{\sim}{r}_0)$ given prices $(\underset{\sim}{w}_0, \underset{\sim}{r}_0)$, for some initial condition, x_0, where $x_t = (s_t, n_{t-1})$.

[45] Details on the necessary utility transformations that lead to this result are available upon request.

Proposition S.1 *There is a unique function* $U : \widetilde{b}(.) \times \mathbb{X} \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \to \mathbb{R}$ *satisfying equation (4).*

Proof. We prove this proposition in three main steps and several intermediate steps. For full details, see Technical Appendix T.1.
Let $\mathbb{X}_D \equiv \{\underset{\sim}{x}_{D0} : \underset{\sim}{x}_{D0} \in \Pi_D(x_{D0}; \underset{\sim}{w}, \underset{\sim}{r}),$ some $x_{D0} \in \mathbb{R}_+^3, ||\underset{\sim}{x}_{D0}||_\kappa^\nu < \infty\}$,
where $||\underset{\sim}{x}_{D0}||_\kappa^\nu \equiv [\sup\{(||x_{Dt}||_E/\kappa^t)^\nu : t \geq 0\}]^{1/\nu}$,
$\mathbb{U}_D \equiv \{\mathcal{U} : \mathbb{X}_D \times \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R} : \mathcal{U}$ continuous, h.o.d. ν, and ${}_\nu||\mathcal{U}|| < \infty\}$,
where ${}_\nu||\mathcal{U}|| \equiv \sup\{|\mathcal{U}(\underset{\sim}{x}_D)|/||(\underset{\sim}{x}_D)||_\kappa^\nu : \underset{\sim}{x}_D \in \mathbb{X}_D \subseteq \mathbb{R}^\infty\}$

Step 1: If Assumption S.2 (which is equivalent to Assumption 1) is satisfied, the function U_{Dt}^o is uniquely defined through equation (S.10). This part closely follows the proof of Lemma 5, p. 37, in Alvarez (1994).
To prove Step 1, we proceed as follows:

- Given $\mathcal{U} \in \mathbb{U}_D$, define an operator $\mathcal{J}\mathcal{U}$ from the right-hand side of equation (S.10);
- show that $\mathbb{U}_D$ is complete;
- show that $\mathcal{J}$ maps $\mathbb{U}_D$ into itself;
- show that $\mathcal{J}$ is a contraction of modulus $\kappa^\nu\zeta$.

  To show that $\mathcal{J}$ is a contraction mapping, we follow Alvarez (1994) using an appropriately modified version of Blackwell's sufficient conditions (see Lemma T.1). The only difference from the standard version is the definition of $(\mathcal{U} + a)$. Given this, we need to show that:

  - $\mathcal{J}$ is monotone;
  - $\mathcal{J}$ discounts.

Step 2: Using the definition of U_{Dt}^o in relation to U_t^o in equation (S.5), it is easy to see that U_t^o is uniquely defined through equation (S.3).
To prove Step 2, we show:

- *Claim i:* U^o solves equation (S.3);
- *Claim ii:* U^o is unique.

Step 3: Using the definition of U_t^o in relation to U_t^m in equation (S.4), it follows that $U_t = U_t^m$ is uniquely defined through equation (4) in the main text.
To prove Step 3, we show:

- *Claim iii:* U^m solves equation (4);
- *Claim iv:* U^m is unique.

■

S.4 Characterization of equilibrium allocations: details

We have not yet fully characterized the equilibrium. To do this, we show equivalence of Definition S.2 to a planning problem in sequence form which has a natural recursive formulation, so that we can apply standard dynamic programming techniques (Section S.4.1). Note that because of the transfer constraint, this will not be a social planner in the usual sense. Rather, we write down a modified planning problem where the constraint is explicitly taken into account. We call it a "Pseudo-Planner's problem," U_P^*. The "Pseudo-Planner" takes the marginal product of labor and the minimum transfer constraint as given in the intra-temporal allocation of consumption, but is otherwise only subject to feasibility. Note that thanks to Assumption 2, the Pseudo-Planner's constraint set is convex. We then show that, thanks to Assumption S.2, her value function is increasing, concave and differentiable so that first-order and envelope conditions can be derived. It is then fairly standard to show that the intra-temporal transfer condition, the Euler equations together with the transversality condition are necessary and sufficient to characterize the Pseudo-planner's optimum (Section S.4.2). Rewriting them in terms of household variables, using the firm's optimality conditions and the budget constraints gives us the equations above. Because of the equivalences described in Sections S.1 and S.2 and Lemma S.2 below, this shows that equations (6) to (11) together with the budget constraints and feasibility are necessary and sufficient to characterize the equilibrium. Finally, in Section S.4.4, we use the Pseudo-Planner's problem to show that decision rules and equilibrium prices are continuous functions on $\underline{b}$. This is needed in several of the proofs in the main text (see results in Section 4.2).[46]

S.4.1 Pseudo-Planner's Problem: Setup and Properties

Let us first set up the Pseudo-Planner's problem in sequence form where the Pseudo-Planner takes the numbers M_{Pt} as given. In Lemma S.2, M_{Pt} will be set to the marginal

[46]See also Figure S.1 for a graphical depiction of the results in this appendix.

product of labor which leads to the equivalence to the dynastic sequence problem and the equilibrium in Section 3.1.3.

Let us define the optimal transfer function, $\widetilde{B}_P$, $\forall t \geq 0, z \in \mathbb{R}$:

$$\widetilde{B}_P(z, x_{Pt}, x_{Pt+1}; M_{Pt}) \equiv \tag{S.17}$$
$$\underset{B_t \geq N_t^m \underline{b}_t}{argmax}\{\beta u_D(F(K_t, N_t^m) - (N_t^m + B_t)M_{Pt}, N_t^o) + \Psi_D(N_t^m, u_D(M_{Pt}(N_t^m + B_t) - K_{t+1} - \theta_t N_{t+1}^m, N_t^m) + z)$$

where $x_{Pt} = (K_t, N_t^m, N_t^o)$. Note that given Assumption S.2(c), we have

$$\widetilde{B}_P(\lambda^\nu z, \lambda x_{Pt}, \lambda x_{Pt+1}; M_{Pt}) = \lambda \widetilde{B}_P(z, x_{Pt}, x_{Pt+1}; M_{Pt}). \tag{S.18}$$

Note that this homogeneity property will help in the proofs. In particular, it will be used to show that the value function is homogeneous of degree ν.

Let the set of feasible allocations from $x_{Pt} = (K_t, N_t^m, N_t^o)$ be given by

$$\Pi_P(x_{Pt}) \equiv \{\underset{\sim}{x}_{Pt} : K_{k+1} + \theta_k N_{k+1}^m \leq F(K_k, N_k^m), \forall k \geq t, x_{Pt} \text{ given}\}.$$

Hence, the Planner's utility as a function of the infinite sequence of state variables $\underset{\sim}{x}_{Pt} \in \Pi_P(x_{Pt})$, and $\underset{\sim}{M}_{Pt}$, can be written as

$$\begin{aligned} U_P^o(\underset{\sim}{x}_{Pt}; \underset{\sim}{M}_{Pt}) \quad = \quad & \beta u_D(F(K_t, N_t^m) - (N_t^m + \widetilde{B}_P(.))M_{Pt}, N_t^o) \\ & + \Psi_D(N_t^m, u_D(M_{Pt}(N_t^m + \widetilde{B}_P(.)) - K_{t+1} - \theta_t N_{t+1}^m, N_t^m) + U_P^o(\underset{\sim}{x}_{Pt+1}; \underset{\sim}{M}_{Pt+1})) \end{aligned} \tag{S.19}$$

where $\underset{\sim}{x}_{Pt} = (x_{Pt}, \underset{\sim}{x}_{Pt+1})$, i.e. $\underset{\sim}{x}_{Pt+1}$ agrees with $\underset{\sim}{x}_{Pt}$ from $t+1$ on. Note that a version of the *Claim* in Proposition S.1 can be used to show that U_P^o is uniquely defined through equation (S.19).

Definition S.4 *Pseudo-Planner's Sequence Problem (PSP):*
$U_P^{o**}(x_{P0}; \underset{\sim}{M}_{P0}) = \sup\{U_P^o(\underset{\sim}{x}_{P0}; \underset{\sim}{M}_{P0}) : \underset{\sim}{x}_{P0} \in \Pi_P(x_{P0})\}$

Lemma S.2 *Let $M_{Pt} = F_N(K_t, N_t^m)$, $\forall t$. Then $\widetilde{B}_P$ from equation (S.17) is equal to $\widetilde{B}$ from equation (S.8), the solution to the Pseudo-planner's problem in Definition S.4 corresponds to the solution to the dynastic problem in Definition S.2.*

Proof. By Euler's theorem, we have $F_K K = F(K, N) - F_N N$. Using this and the firm's optimality conditions, the objective in equations (S.8) and (S.17) are the same and so is the constraint, $B_t \geq N_t^m \underline{b}$. Hence $\widetilde{B}_P = \widetilde{B}$. Euler's theorem also implies that the constraint set for the Pseudo-planner, Π_P, is the same as for the dynasty, Π_D. Again, using Euler's theorem and the firm's optimality conditions, the arguments of u_D and Ψ_D in equations (S.19) and (S.10) are the same. Hence, the utility recursively defined through these equations are the same. Therefore, the solutions to the two problems are the same. ■

Thanks to the equivalences in Sections S.1 and S.2, written in terms of household variables, the allocation is also equivalent to the equilibrium allocation described in Section 3.1.3. Thus, if we find necessary and sufficient conditions to characterize the Pseudo-Planner's problem, we can rewrite them in terms of household variables and obtain a set of necessary and sufficient conditions to characterize our equilibrium.

To do so, we first need to show that the Pseudo-Planner's value function has the desired properties. This is most easily done in a recursive formulation of the problem.

Definition S.5 *Pseudo-Planner's FE (PFE): Given a stationary law of motion for $M_P' = L_P(M_P)$, define the following Bellman equation:*

$$U_P^*(x_P; M_P) = \sup_{x_P'}\{\beta u_D(F(K, N^m) - (N^m + \widetilde{B}_P(.))M_P, N^o) + \Psi_D[N^m, u_D(M_P(N^m + \widetilde{B}_P(.) - K' - \theta N^{m'}), N^m) + U_P^*(x_P', M_P'))]$$
$$x_P' \text{ s.t. } [K' + \theta N^{m'} \leq F(K, N^m)]\}.$$

Let $\mathscr{C}_P(\nu) \equiv \{F : X_P \times \mathbb{R}_+ \to \mathbb{R} : F$ continuous, F h.o.d. $\nu, ||F||_\nu < \infty\}$
where $X_P \subseteq \mathbb{R}^3_+$ is the state space and $||F||_\nu \equiv \sup\{(|F(x_P)|/(||x_P||_E)^\nu : x_P \in X_P\}$.
Further, define the operator, T_P, as:

$$(T_P U_P)(x_P; M_P) = \sup_{x_P'}\{\beta u_D(.,.) + \Psi_D[N^m, u_D(.,.) + U_P(x_P'; M_P'))]$$
$$x_P' \text{ s.t. } [K' + \theta N^{m'} \leq F(K, N^m)]\}.$$

The following Lemma states the standard dynamic programming result that T_P has a unique fixed point, U_P^{o*}, which corresponds to the maximized value of the dynastic sequence problem U_P^{o**} in Definition S.4.

Lemma S.3 *If Assumption S.2 holds and $\kappa^\nu\zeta < 1$, then $T_P : \mathscr{C}_P(\nu) \to \mathscr{C}_P(\nu)$, $\mathscr{C}_P(\nu)$ is a Banach space; T_P is a contraction of modulus $\kappa^\nu\zeta < 1$ in $\mathscr{C}_P(\nu)$; there is a unique $U_P^{o*} \in \mathscr{C}_D(\nu)$, such that $U_P^{o*} = T_P U_P^{o*}$; if $\underset{\sim}{M}_P$ agrees with M_P and $L_P(.)$, then $(U_P^{o*} = T_P U_P^{o*})$ implies $(U_P^{o*} = U_P^{o**})$; and the associated policy correspondences $K'(x_P; M_P), N^{m'}(x_P; M_P)$ are non-empty, h.o.d. 1 in x_P and u.h.c..*

The proof of Lemma S.3 is standard, except for the definition of "adding a constant to a function" and the version of Blackwell's sufficient conditions used.[47] Next, we derive important properties for the value function, U_P^*.

Lemma S.4 *If Assumption S.2 is satisfied, then U_P^* is strictly increasing, strictly concave, differentiable and homogeneous of degree ν in x_P, and the associated policy correspondences are single valued, continuous and homogeneous of degree 1.*

Proof. For monotonicity, see Stokey, Lucas, and Prescott (1989), Theorem 4.7, p.80, and for strict concavity, see Stokey, Lucas, and Prescott (1989), Theorem 4.8, p.81. Stated briefly, since the constraint set is a convex cone and monotone and since u_D is strictly increasing and strictly concave while Ψ_D is strictly increasing and weakly concave, the proofs go through exactly as long as the norm $||.||_\nu$ defined above and boundedness condition in Assumption S.3 are used. To show differentiability is slightly more involved but, as usual, the goal is to apply Benveniste-Scheinkman's Envelope Theorem (see Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84). First, the definition of $\widetilde{B}_P$ in equation (S.17) implies the following intra-temporal FOC (using only the differentiability of u_D and Ψ_D):

$$\beta \frac{\partial u_D(C^o, N^o)}{\partial C^o} = \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U_P^*(x_P^{'*}))}{\partial U} \frac{\partial u_D(C^m, N^m)}{\partial C^m} + \lambda_B / M \quad \text{(S.20)}$$

where λ_B is the multiplier on the minimum transfer constraint. Note that $U_D^*(x_D^{'*})$ and $U_P^*(x_P^{'*})$ are just numbers here.

There are thus three cases: $(x_P, x_P^{'}(x_P))$ is such that (1) $\widetilde{B}_P > \underline{b}N^m$ and $\lambda_B = 0$, (2) $\widetilde{B}_P = \underline{b}N^m$ and $\lambda_B > 0$, (3) $\widetilde{B}_P = \underline{b}N^m$ and $\lambda_B = 0$.

For differentiability with respect to K and N^o, we can always construct a strictly concave, differentiable function, G, that is equal to U_P^* at the current state and below in a small interval around the optimal choice. The reason why this function can be strictly concave and differentiable is because u_D and Ψ_D are (strictly/weakly) concave and differentiable by assumption. Since the minimum transfer constraint is unaffected by K and N^o, we can hold B constant on this interval. The reason why for every point around the current state the optimal choice for tomorrow's state is still feasible is because $C^{m*} > 0$ and $C^{o*} > 0$ since $u_D'(0, N) = \infty$. That is, we can adjust consumption to accommodate the lower current state and still achieve the optimal choice for tomorrow.

[47] See Proposition 4' in Alvarez (1994)), p. 40-41 and p. 65-67. The proof of this proposition used Proposition 3, p. 26, and the proof of Proposition 4, p. 27.

Given this, we can apply Benveniste-Scheinkman to deduce that U_P^* is differentiable in K and N^o.

For differentiability with respect to N^m, the argument is slightly more involved. In cases (1) and (2), we can use the same argument as above, except that we have to adjust B in case (2) which is possible since consumption is strictly positive at an optimum as above. So U_P^* is differentiable in the set of x_P such that (1) or (2) holds. In case (3), decreasing N^m brings us to case (1) and increasing N^m brings us to case (2). Hence, we have to show that the derivatives in cases (1) and (2) are equal as we approach case (3). Indeed, the only difference between the derivatives in case (1) and (2) is λ_B, which approaches 0 as we approach case (3) from case (2). Hence, U_P^* is differentiable with respect to N^m. See Technical Appendix T.4 for details.

That U_P^* is homogeneous of degree ν was shown in Lemma S.3. The properties of the decision rules follow in the usual way from the properties of the value function. ■

S.4.2 Necessary and Sufficient Conditions for Optimality

Proposition S.2 *If $M_P = F_N$, any sequence $\underset{\sim}{x}_{P0} \in \Pi_P(x_{P0})$ is optimal for the problem in Definition S.4 if and only if it satisfies the intra-temporal allocation condition (S.21), the Euler equations (S.22) and the transversality condition (S.23).*
Intra-temporal allocation condition:

$$B: \quad \beta \frac{\partial u_D(C_t^o, N_t^o)}{\partial C_t^o} = \frac{\partial \Psi_D(N_t^m, u_D(C_t^m, N_t^m) + U_{Pt+1}^*)}{\partial U} \frac{\partial u_D(C_t^m, N_t^m)}{\partial C_t^m} + \lambda_{B,t}/F_{Nt} \tag{S.21}$$

Euler Equations:

$$K': \quad \frac{\partial u_D(C_t^m, N_t^m)}{\partial C_t^m} = \beta \frac{\partial u_D(C_{t+1}^o, N_{t+1}^o)}{\partial C_{t+1}^o} F_{Kt+1} \tag{S.22}$$

$$\begin{aligned} N^{m'}: \quad \theta \frac{\partial u_D(C_t^m, N_t^m)}{\partial C_t^m} &= \frac{\partial \Psi_D(N_{t+1}^m, u_D(C_{t+1}^m, N_{t+1}^m) + U_{Pt+2}^*)}{\partial N_{t+1}^m} \\ &+ \frac{\partial \Psi_D(N_{t+1}^m, u_D(C_{t+1}^m, N_{t+1}^m) + U_{Pt+1}^{*'})}{\partial U} \\ &\quad * \left[F_{Nt+1} \frac{\partial u_D(C_{t+1}^m, N_{t+1}^m)}{\partial C_{t+1}^m} + \frac{\partial u_D(C_{t+1}^m, N_{t+1}^m)}{\partial N_{t+1}^m} + \beta \frac{\partial u_D(C_{t+2}^o, N_{t+2}^o)}{\partial N_{t+2}^o} \right] \\ &- \lambda_{B,t+1} \underline{b}_{t+1} \end{aligned}$$

Transversality condition

$$\zeta^t \left[\frac{\partial u_D(C_t^{o*}, N_t^o)}{\partial C_t^{o*}} F_{Kt}(K_t + \theta N_t^m) + \frac{\partial u_D(C_t^{o*}, N_t^o)}{\partial N_t^o} N_t^o \right] \to 0 \text{ as } t \to \infty \tag{S.23}$$

Proof. For necessity of equations (S.21) to (S.23), we closely follow the arguments in Alvarez (1994), Proposition 8. That is, suppose an allocation attains the sup in Definition S.4, $U_P^{**}(x_{P0})$ where $U_P^{**} = U_P^*$ by Lemma S.3 and where $\widetilde{B}_P(.)$ is defined in equation (S.17). Then, by Lemma S.4, the first-order and envelope conditions can be derived which lead to equations (S.21) and (S.22). To show that the transversality condition must hold at an optimum, we first show that $\zeta^t U_P^{o*}(\underset{\sim}{x}_{Pt}) \to 0$ as $t \to \infty$. Using the fact that U_P^{o*} is h.o.d. ν and the envelope conditions and Euler equations, (S.23) follows. For sufficiency of equations (S.21) to (S.23), we closely follow the proof of Theorem 4.15 in Stokey, Lucas, and Prescott (1989), p. 98. The main difference is that due to the recursively defined utility, we cannot write it as an infinite sum. However, thanks to Assumption S.2(d) we show that $\frac{\partial \Psi_D(N^m, U)}{\partial U} \leq \zeta < 1$ and since $\zeta^t U_P^{o*}(\underset{\sim}{x}_{Pt}) \to 0$ as $t \to \infty$, the result follows. For full details see Technical Appendix T.2. ■

Expressing the conditions in Proposition S.2 in per capita terms, using the relationship between u_D and u and Ψ_D and Ψ in equations (S.5), and the firm's optimality conditions, we get the necessary and sufficient conditions of the household problem in the text (see Section 3.2). For algebra details see Technical Appendix T.3.

S.4.3 In an unconstrained equilibrium, $r_{t+1}\theta_t > w_{t+1}$

Consider the optimality conditions in equations (S.21) and (S.22). If the equilibrium is unconstrained, then $\lambda_{B,t+1} = 0$. Substituting equation (S.21) and the first Euler equation in (S.22) into the second Euler equation in (S.22), substituting marginal products for prices and rearranging terms, we get:

$$\frac{\partial \Psi_D(.)}{\partial N_{t+1}^{m*}} + \frac{\partial \Psi_D(.)}{\partial U} \left(\frac{\partial u_D(C_{t+1}^{m*}, N_{t+1}^{m*})}{\partial N_{t+1}^{m*}} + \frac{\partial u_D(C_{t+2}^{o*}, N_{t+2}^{o*})}{\partial N_{t+2}^{o*}} \right) = \frac{\partial u_D(C_{t+1}^{o*}, N_{t+1}^{o*})}{\partial C_{t+1}^{o*}} (r_{t+1}\theta_t - w_{t+1}).$$

The left-hand side is the marginal utility benefit of an additional child to the dynasty. That is strictly positive by Assumption S.2. Hence, the right-hand side must be positive in equilibrium. Since u_D is strictly increasing, we have that $r_{t+1}\theta_t > w_{t+1}$ in an unconstrained equilibrium.

S.4.4 Equilibrium decision rules and prices are continuous in $\underline{b}$

In this section, we show that household decision rules and prices are continuous in the minimum transfer constraint, $\underline{b}$. This result will be used in the proof of Propositions 4 and 6 below.

Proposition S.3 *U_P^* is differentiable with respect to the parameter $\underline{b}$ and the Pseudo Planner's policy functions are continuous in $\underline{b}$. Therefore, household policy functions and equilibrium prices are continuous in $\underline{b}$.*

Proof. Add $\underline{b}$ as a state variable with law of motion $\underline{b}' = \underline{b}$. We want to show that $U_P^*(x_P; \underline{b}, M_P)$ differentiable in $\underline{b}$. As usual, the goal is to apply Benveniste-Scheinkman's Envelope Theorem (see Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84). To do so, define $\underline{b}^* \in [-1, \underline{b}^{max}]$ such that, given (K_P, N_P^m, N_P^o), we have $B = N_P^m \underline{b}^*$ and

$$\beta \frac{\partial u_D(o, x_P, \underline{b}^*)}{\partial C^o} = \frac{\partial \Psi_D(N_P^m, u_D(m, x_P, \underline{b}^*) + U_P^{o*}(x_P'(x_P, \underline{b}^*)))}{\partial U} \frac{\partial u_D(m, x_P, \underline{b}^*)}{\partial C^m}.$$

That is, the constraint just starts to bind at $\underline{b}^*$ but $\lambda_B = 0$.[48] There are three cases: given $(x_P, x_P'(x_P, \underline{b}))$ (i) $\underline{b} < \underline{b}^*$ and $\lambda_B = 0$, (ii) $\underline{b} > \underline{b}^*$ and $\lambda_B > 0$, (iii) $\underline{b} = \underline{b}^*$ and $\lambda_B = 0$.

In case (i), we can always construct a strictly concave, differentiable function, G, that is equal to U_P^* at $\underline{b}$ and below in a small interval around it. The reason why this function can be strictly concave and differentiable is because u_D and Ψ_D are (strictly/weakly) concave and differentiable by assumption. Since the minimum transfer constraint is still not binding in a small enough interval, we can hold B and the optimal choice for tomorrow's state at their values for $\underline{b}$ on this interval without violating any budget constraints. Given this, we can apply Benveniste-Scheinkman to deduce that U_P^* is differentiable in $\underline{b}$ in case (i) with $\frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|i} = 0$.

In case (ii), we can use the same argument as above, except that we have to adjust B. The reason why for every point in a small enough interval around $\underline{b}$ the optimal choice for tomorrow's state is still feasible is because $C^{m*} > 0$ and $C^{o*} > 0$ since $u_D'(0, N) = \infty$. That is, we can adjust consumption to accommodate the higher minimum transfer constraint and still achieve the optimal choice for tomorrow. So U_P^* is differentiable in case (ii) with $\frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|ii} = -M_P N_P^m \lambda_B$.

In case (iii), decreasing $\underline{b}$ brings us to case (i) and increasing $\underline{b}$ brings us to case (ii). Hence, we have to show that the derivatives in cases (i) and (ii) are equal as we approach case (iii). Indeed, λ_B approaches 0 as we approach case (iii) from case (ii).

[48] Note that $\underline{b}^*$ exists if $\partial \Psi_D / \partial U > 0$. If $\partial \Psi_D / \partial U = 0$, only Case (ii) below is relevant.

Hence, U_P^* is differentiable with respect to $\underline{b}$. Thus U_P^* is continuous in $\underline{b}$ and by the Theorem of the Maximum, the optimal policy functions, x'_P are continuous in $\underline{b}$. Using the relationships $s_{t+1} = K_{Pt+1}/N_{Pt}^m$ and $n_t = N_{Pt+1}^m/N_{Pt}^m$, it follows that household policy functions are also continuous in $\underline{b}$. Since F is continuously differentiable, $F_K(.)$ and $F_N(.)$ are continuous functions of x_P. Therefore, equilibrium prices, $w = F_N(x_P(\underline{b}))$ and $r = F_K(x_P(\underline{b}))$, are also continuous in $\underline{b}$. See Technical Appendix T.5 for details. ■

S.5 Recursive Competitive Equilibrium

The equilibrium described in Section 3.1.3 can also be written as a recursive competitive equilibrium (RCE). For some applications, this formulation may be more intuitive and convenient. We report it here for completeness: definition, equivalence result and relation to the Pseudo-Planner's problem.

Let $X \subseteq \mathbb{R}_+^2$ be the household state space and let $X_D \subseteq \mathbb{R}_+^3$ be the dynastic or aggregate state space.

Definition S.6 *A Recursive Competitive Equilibrium (RCE) is a collection of:*

- *continuous price functions:* $w : X_D \to \mathbb{R}_{++}$ *and* $r : X_D \to \mathbb{R}_{++}$,
- *value functions:* $U^{o*} : X \times X_D \to \mathbb{R}$ *and* $U^{m*} : X \times X_D \to \mathbb{R}$,
- *household decision functions:* $c^{o*} : X \times X_D \to \mathbb{R}_+$, $b^* : X \times X_D \to [\underline{b}, \overline{b}]$,

  $c^{m*} : [\underline{b}, \overline{b}] \times X_D \to \mathbb{R}_+$, $s^{*'} : [\underline{b}, \overline{b}] \times X_D \to \mathbb{R}_+$, $n^{*'} : [\underline{b}, \overline{b}] \times X_D \to \mathbb{R}_+$,
- *firm decision rules:* $\widetilde{K}^* : X_D \times X_D \to \mathbb{R}_+$, $\widetilde{N}^* : X_D \times X_D \to \mathbb{R}_+$;
- *laws of motion for the aggregate state variables:* $L : X_D \to X_D$,

such that for all $(x, \widehat{x}_D) \in X \times X_D$:

- *given* $w(\widehat{x}_D), r(\widehat{x}_D)$ *and* b, *the decision rules* $c^{m*}(b, \widehat{x}_D)$, $s^{*'}(b, \widehat{x}_D)$ *and* $n^{*'}(b, \widehat{x}_D)$ *maximize* $u(c^m) + U^{o*}(s', n'; \widehat{x}'_D)$, *subject to* $c^m \leq w(\widehat{x}_D)(1 + b) - \theta n' - s'$
- *given* $w(\widehat{x}_D), r(\widehat{x}_D)$, (s, n), *the decision rules* $c^{o*}(s, n; \widehat{x}_D)$ *and* $b^*(s, n; \widehat{x}_D)$ *maximize* $\beta u(c^o) + \Psi(n, U^{m*}(b; \widehat{x}_D))$, *subject to* $c^o \leq r(\widehat{x}_D)s - w(\widehat{x}_D)nb$
- $U^{m*}(b, \widehat{x}_D) = u(c^{m*}(b, \widehat{x}_D)) + U^{o*}(s^{*'}(b, \widehat{x}_D), n^{*'}(b, \widehat{x}_D); \widehat{x}'_D)$

  $U^{o*}(s, n; \widehat{x}_D) = \beta u(c^{o*}(s, n; \widehat{x}_D)) + \Psi(n, U^{m*}(b^*(s, n; \widehat{x}_D); \widehat{x}_D))$

- *given* $w(\widehat{x}_D), r(\widehat{x}_D)$*, decision rules* $\widetilde{K}^*(\widehat{x}_D)$ *and* $\widetilde{N}^*(\widehat{x}_D)$ *maximize* $F(\widetilde{K},\widetilde{N}) - w(\widehat{x}_D)\widetilde{N} - r(\widehat{x}_D)\widetilde{K}$*,*

- *markets clear:*

  $$\widehat{N}^m c^{m*}(b^*(\widehat{x},\widehat{x}_D),\widehat{x}_D) + \widehat{N}^o c^{o*}(\widehat{x},\widehat{x}_D) + \widehat{N}^m s^*(b^*(\widehat{x},\widehat{x}_D),\widehat{x}_D) + \theta \widehat{N}^m n^*(b^*(\widehat{x},\widehat{x}_D),\widehat{x}_D)$$
  $$= F(\widetilde{K}(\widehat{x}_D),\widetilde{N}(\widehat{x}_D)),$$
  $$\widetilde{K}^*(\widehat{x}_D) = \widehat{K} \text{ and } \widetilde{N}^*(\widehat{x}_D) = \widehat{N}^m,$$

- *consistency:*

  $$\widehat{x} = (\widehat{s},\widehat{n}) = (\widehat{K}/\widehat{N}^o, \widehat{N}^m/\widehat{N}^o)$$
  $$\widehat{K}' = L_1(\widehat{x}_D) = \widehat{N}^m s^{*'}(b^*(\widehat{x},\widehat{x}_D);\widehat{x}_D)$$
  $$\widehat{N}^{m'} = L_2(\widehat{x}_D) = \widehat{N}^m n^{*'}(b^*(\widehat{x},\widehat{x}_D);\widehat{x}_D)$$
  $$\widehat{N}^{o'} = L_3(\widehat{x}_D) = \widehat{N}^m$$
  $$\widehat{N}^o_0 = 1, \widehat{N}^m_0 = n_0, \widehat{K}_0 = k_0 \text{ given.}$$

The RCE formulation from Definition S.6 has two optimization problems, one for old households and one for middle-aged households. The first order conditions and the envelope condition for the old household problem are

$$\begin{aligned}
b: &\quad \beta u'(c^o(s,n;\widehat{x}_D))nw(\widehat{x}_D) = \Psi_U(n,U^m(b;\widehat{x}_D))\frac{\partial U^m(b;\widehat{x}_D)}{\partial b} + \lambda_b \\
EC_s: &\quad \frac{\partial U^o(s,n;\widehat{x}_D)}{\partial s} = \beta u'(c^o(s,n;\widehat{x}_D))r(\widehat{x}_D) \\
EC_n: &\quad \frac{\partial U^o(s,n;\widehat{x}_D)}{\partial n} = \Psi_n(n,U^{m*}(b;\widehat{x}_D)) - \beta u'(c^o(s,n;\widehat{x}_D))bw(\widehat{x}_D)\ .
\end{aligned}$$

Similarly, from the young household problem we have

$$\begin{aligned}
n: &\quad u'(c^m(b;\widehat{x}_D))\theta = \frac{\partial U^o(s^{*'}(b,\widehat{x}_D), n^{*'}(b,\widehat{x}_D;\widehat{x}_D)}{\partial n} \\
s: &\quad u'(c^m(b;\widehat{x}_D)) = \frac{\partial U^o(s^{*'}(b,\widehat{x}_D), n^{*'}(b,\widehat{x}_D);\widehat{x}_D)}{\partial s} \\
EC_b: &\quad \frac{\partial U^m(b;\widehat{x}_D)}{\partial b} = u'(c^m(b;\widehat{x}_D))w(\widehat{x}_D)\ .
\end{aligned}$$

Using the envelope conditions to substitute out for $\frac{\partial U^m(b;\widehat{x}_D)}{\partial b}$, $\frac{\partial U^o(s,n;\widehat{x}_D)}{\partial s}$ and $\frac{\partial U^o(s,n;\widehat{x}_D)}{\partial n}$, and suppressing the arguments, we can write the necessary equilibrium conditions as:

$$\begin{aligned} n: &\quad u'(c^m)\theta + \beta u'(c^{o'})b'w' = \Psi_n \\ s: &\quad u'(c^m) = \beta u'(c^{o'})r' \\ b: &\quad \beta u'(c^o)n = \Psi_U u'(c^m) + \frac{\lambda_b}{w} \end{aligned}$$

Adding in the time indices, we obtain the FOCs in equations (6) to (8) in the main text.

While intuitive, this does not prove that these equations (together with the firm's optimality conditions and the budget constraints) are sufficient to characterize equilibria. Since there are non-covexities in the constraint sets here, we set up an equivalent Pseudo-Planner's problem in Section S.4 which does not suffer from this problem and derive sufficiency there.

Next, we show that the RCE is equivalent to the equilibrium described in the main text. See also Figure S.1 for a graphical depiction of these equivalences.

Proposition S.4 *The Recursive Competitive Equilibrium is equivalent to the equilibrium defined in Section 3.1.3.*

Outline of Proof. The proof proceeds as follows.
Step 1: From the RCE, we derive the Old Household Functional Equation (HFE, Definition S.7).
Step 2: From HFE, we derive the (equivalent) Dynastic Functional Equation (DFE, Definition S.8).
Step 3: We then show that DFE has a unique fixed point, U_D^{o*} which attains the supremum of the Dynastic sequence problem (DSP) in Definition S.2, U_D^{o**} (see Lemma S.5).
Step 4: Given (2) and (3) and Lemma S.1 (i.e. equivalence Old Household Sequence Problem (HSP) in Definition S.1 and DSP in Definition S.2), we can show that the HFE is equivalent to the HSP. In addition, since HSP was derived from the equilibrium described in Section 3.1.3 and HFE was derived from the RCE, the household problems in Section 3.1.3 and in the RCE are equivalent (see Corollary S.1).
Step 5: It then only remains to point out that the firm's optimality conditions as well as market clearing conditions coincide in the two equilibrium definitions.
Step 6 concludes.
Within this proof, we also state some definitions that will be used in subsequent sections and refer to similar proofs in Alvarez (1994).
Proof.

Step 1: Derive Old Household Functional Equation

Using the same steps as in Section S.1 on U^m and U^o from the RCE, we can write the old household problem as a functional equation by defining the operator, T, as follows.

Definition S.7 *Old Household FE (HFE):*
Given a stationary law of motion for $\widehat{x}'_D = L(\widehat{x}_D)$, define the operator, T, as:

$$(TU^o)(s, n; \widehat{x}) = \sup_{x'}\{\beta u(r(\widehat{x}_D)s - n\widetilde{b}w(\widehat{x}_D)) + \Psi[n, u(w(\widehat{x}_D)(1+\widetilde{b}(.)) - s' - \theta n') + U^o(x'; \widehat{x}'_D)],$$
$$x' \text{ s.t. } [(s' + \theta n')n \leq w(\widehat{x}_D)n + r(\widehat{x}_D)s]\}$$

Step 2: Derive Dynastic Functional Equation

Similarly, using the same steps as in Section S.2 on U^o and U^o_D, we can write the dynastic problem as a functional equation by defining the operator, T_D, as as follows

Definition S.8 *Dynastic FE (DFE): Given a stationary law of motion for $\widehat{x}' = L(\widehat{x})$, define the operator, T_D, as:*

$$(T_D U^o_D)(x_D; \widehat{x}_D) = \sup_{x'_D}\{\beta u_D(r(\widehat{x}_D)S - \widetilde{B}(.)w(\widehat{x}_D), N^o)$$
$$+\Psi_D[N^m, u_D(w(\widehat{x}_D)(N^m + \widetilde{B}(.)) - S' - \theta N^{m'}, N^m) + U^o_D(x'_D, \widehat{x}'_D); \widehat{x}'_D)],$$
$$x'_D \text{ s.t. } [S' + \theta N^{m'} \leq w(\widehat{x}_D)N^m + r(\widehat{x}_D)S]\}$$

Step 3: The problem in Def. S.8 (DFE) is equivalent to the problem Def. S.2 (DSP).

The following Lemma states the standard dynamic programming result that T_D has a unique fixed point, U^{o*}_D, which corresponds to the maximized value of the dynastic sequence problem U^{o**}_D in Definition S.2.

Let $\mathscr{C}_D(\nu) \equiv \{F : X_D \times X_D \to \mathbb{R} : F$ is continuous, F is h.o.d. ν and $||F||_\nu < \infty\}$ where $||F||_\nu \equiv \sup\{(|F(x_D)|/(||x_D||_E)^\nu : x_D \in X_D \subseteq \mathbb{R}^3\}$.

Lemma S.5 *If Assumption S.2 holds and $\kappa^\nu\zeta < 1$, then:*

1. *$T_D : \mathscr{C}_D(\nu) \to \mathscr{C}_D(\nu)$, $\mathscr{C}_D(\nu)$ is a Banach space,*
2. *T_D is a contraction of modulus $\kappa^\nu\zeta < 1$ in $\mathscr{C}_D(\nu)$,*
3. *there is a unique $U^{o*}_D \in \mathscr{C}_D(\nu)$, such that $U^{o*}_D = T_D U^{o*}_D$,*
4. *if $(\underset{\sim}{w}, \underset{\sim}{r})$ agrees with $(w(\widehat{x}), r(\widehat{x}))$, then*
   $U^{o*}_D = T_D U^{o*}_D$ implies $U^{o*}_D = U^{o**}_D$.

5. *And the associated policy correspondences* $S'(x_D; \widehat{x}_D), N^{m'}(x_D; \widehat{x}_D)$ *are non-empty, h.o.d. 1 in* x_D *and u.h.c..*

The proof of Lemma S.5 is standard.[49]

Step 4: The problem in Def. S.8 (DFE) is equivalent to the problem in Def. S.7 (HFE). The next Corollary states that T has a unique fixed point, U^{o*}, which corresponds to the maximized value of the old household sequence problem, U^{o**}, in Definition S.1. Let $\mathscr{C}(\nu) \equiv \{f : X \to \mathbb{R} : f \text{ is continuous and } F(x,1) \equiv f(x), ||F||_\nu < \infty\}$ where $||F||_\nu \equiv \sup\{(|F(x,1)|/(||(x,1)||_E)^\nu : x \in X \subseteq \mathbb{R}^2\}$.

Corollary S.1 *If Assumption S.2 holds,* $\kappa > 0$ *and* $\kappa^\nu \zeta < 1$*, then*

1. $T : \mathscr{C}(\nu) \to \mathscr{C}(\nu)$, $\mathscr{C}(\nu)$ *is a Banach space,*

2. T *is a contraction in* $\mathscr{C}(\nu)$*,*

3. *there is a unique* $U^{o*} \in \mathscr{C}(\nu)$*, such that* $U^{o*} = TU^{o*}$*,*

4. $U^{o*} = TU^{o*}$ *implies* $U^{o*} = U^{o**}$*.*

5. *And the associated policy correspondences* $s^*(x; \widehat{x}_D), n^*(x; \widehat{x}_D)$ *are non-empty, h.o.d. 1 in* x *and u.h.c..*

The proof of Corollary S.1 is simple once one has shown the equivalence of T and T_D. This can be done very similarly to Step 2 in the proof of Proposition S.1.[50]

Step 5: The firm's optimality conditions and the market clearing conditions are the same in the definition of equilibrium in Section 3.1.3 and in the definition of the RCE.

Step 6: Equivalence between equilibrium in Section 3.1.3 and the RCE.
From Step 4, since HSP was derived from the equilibrium described in Section 3.1.3 and HFE was derived from the RCE, the household problems in Section 3.1.3 and in the RCE are equivalent. From Step 5, the firms optimality conditions as well as market clearing conditions coincide in the two equilibrium definitions. Hence, the equilibria are equivalent. ■

To solve for the RCE, one can use the solution to the Pseudo-Planner's problem. The next Proposition states this relationship:

[49] See Proposition 4' in Alvarez (1994)), p. 40-41 and p. 65-67.

[50] See Propositions 5, 5' and Corollary in Alvarez (1994), p.41 and p. 61-62.

Proposition S.5 *If* $x_P^*(x_P) = (K_P^*(x_P), N_P^{m*}(x_P), N_P^{m*}(x_P))$ *solves the Pseudo Planner's problem with* $L_p(M_P(x_P)) = F_N(L_1(x_P), L_2(x_P))$ *where* $M_P(x_P) = F_N(K, N^m)$ *and letting* $\widehat{x} = x_P$, $x = (K_P, N_P^m)/N^o$, $b = B/N^m$, *where* $B = \widetilde{B}(.)$, *then*

1. *the RCE allocation is given by:*

   - $w(\widehat{x}) = F_N(x_P), r(\widehat{x}) = F_K(x_P),$
   - $c^o(X;\widehat{x}) = [F(K, N^m) - (N^m + \widetilde{B}_P)]/N_P^o, c^m(b;\widehat{x}) = \widetilde{C}_P^m(x_P)/N^m,$
     $s^*(b;\widehat{x}) = K_P^*(x_P)/N_P^m, n^*(b;\widehat{x}) = N_P^{m*}(x_P)/N_P^m,$
   - $U^{o*}(X;\widehat{x}) = U_P^*(x_P, F_N(x_P))/(N^o)^\nu,$
     $U^{m*}(b;\widehat{x}) = u(c^m(b;\widehat{x})) + U^{o*}(x_P^*(x_P)/N^m, x_P^*(x_P)),$
   - $\widetilde{K}(\widehat{x}) = K_P, \widetilde{N}(\widehat{x}) = N_P^m,$
   - $L_1(\widehat{x}) = K_P^*(x_P), L_2(\widehat{x}) = N_P^{m*}(x_P), L_3(\widehat{x}) = N_P^m$;

2. *since* F *is continuously differentiable,* $F_K(.)$ *and* $F_N(.)$ *are continuous functions of* x_P. *Hence, we have found prices* $w(.)$ *and* $r(.)$ *which are continuous functions of* $\widehat{x}$, *a result we use in Section S.4.4 and Appendix A in the main text;*

3. *equations (S.21), (S.22) and (S.23) correspond to equations (6) to (8) and (11) in the main text and—together with budget constraints and firm's optimality conditions—the latter are necessary and sufficient to characterize the equilibrium.*

Proof. Use equivalences in the proof of Proposition S.4 and Lemma S.2. ∎

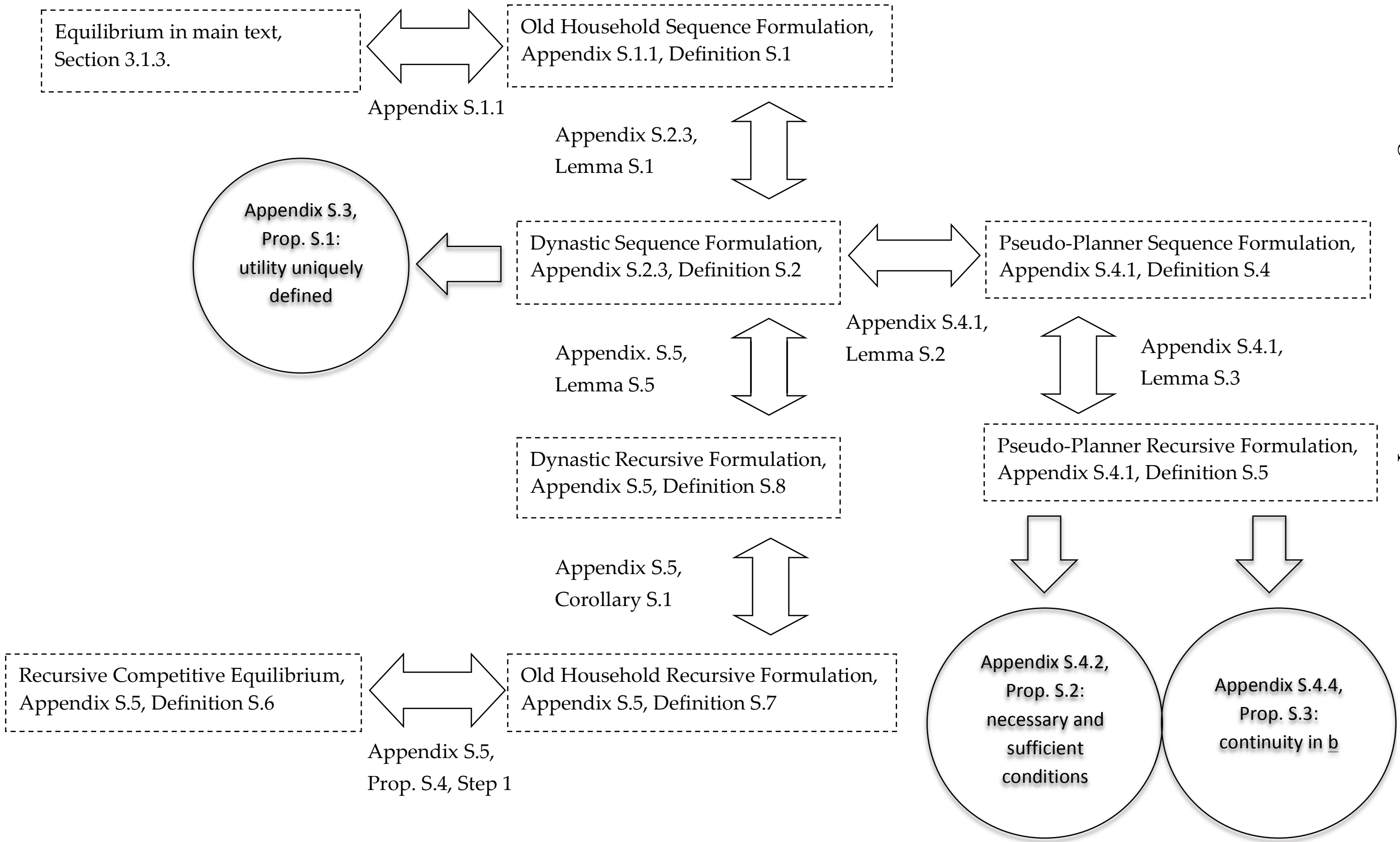

Figure S.1: Formulations, Equivalences and Results