Technical Appendix (online only) to "Property Rights and Efficiency in OLG Models with Endogenous Fertility"

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Abstract

In these notes, we provide very detailed proofs and algebra of some of the results in the Supplementary Appendix to the paper.

Contents

T.1 Detailed proof of Proposition S.1

Proposition:

There is a unique function $U : \widetilde{b}(.) \times \mathbb{X} \times \mathbb{R}_+^\infty \times \mathbb{R}_+^\infty \to \mathbb{R}$ satisfying equation (4).

Proof.

Step 1: If Assumptions S.2 and the boundedness condition in Assumption S.3 are satisfied and if $\kappa^\nu \zeta < 1$, there exists a unique function $U_D^o \in \mathbb{U}_D$ that solves equation (S.10).

Proof of Claim: Assumptions S.2 and S.3 and $\kappa^\nu \zeta < 0$ ensure that we can apply Lemma 5 in Alvarez (1994) (p.37).

- That is, given $\mathcal{U} \in \mathbb{U}_D$, define the operator $\mathcal{J}\mathcal{U}$ by

  $$(\mathcal{J}\mathcal{U})(.) \equiv \beta u_D(r_t S_t - \widetilde{B}(.)w_t, N_t^m)$$
  $$+\Psi_D(N_t^m, u_D(w_t(N_t^m+\widetilde{B}(.))-S_{t+1}-\theta_t N_{t+1}^m, N_t^m)+\mathcal{U}(\underset{\sim}{x}_{Dt+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1})).$$

- $\mathbb{U}_D$ is complete, i.e. given $\{\mathcal{U}_i\}$, a Cauchy sequence with $\mathcal{U}_i \in \mathbb{U}_D, \exists \mathcal{U} \in \mathbb{U}_D$ such that ${}_\nu||\mathcal{U}_i - \mathcal{U}|| \to 0$ as $i \to \infty$.

  To see this, first define $\mathcal{U}$, then show that $\mathcal{U} \in \mathbb{U}_D$.

  - Let $\mathcal{U} \equiv \lim \mathcal{U}_i$ pointwise, which is well-defined since $\{\mathcal{U}_i\}$ is Cauchy.
    Since $\mathcal{U}_i(\underset{\sim}{x}_D) \to \mathcal{U}(\underset{\sim}{x}_D), \forall \underset{\sim}{x}_D \in \Pi_D$, given δ choose i big enough so that
    $\forall \underset{\sim}{x}_D \in \Pi_D, |\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}(\underset{\sim}{x}_D)|/||\underset{\sim}{x}_D||_\kappa^\nu \leq \delta$.
    Hence, ${}_\nu||\mathcal{U}_i - \mathcal{U}|| = sup\{|\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}(\underset{\sim}{x}_D)|/||\underset{\sim}{x}_D||_\kappa^\nu, \underset{\sim}{x}_D \in \Pi_D\} \to 0$ as $i \to \infty$.
  - $\mathcal{U} \in \mathbb{U}_D$:
    * ${}_\nu||\mathcal{U}|| < \infty$:
      given δ, choose i large enough so that
      $\forall \underset{\sim}{x}_D \in \Pi_D, |\mathcal{U}(\underset{\sim}{x}_D)| \leq |\mathcal{U}_i(\underset{\sim}{x}_D)| + \delta||\underset{\sim}{x}_D||_\kappa^\nu$.
      Since $\mathcal{U}_i \in \mathbb{U}_D$, let $B_i \equiv {}_\nu||\mathcal{U}_i|| < \infty$ and note that
      $\forall \underset{\sim}{x}_D \in \Pi_D, |\mathcal{U}_i(\underset{\sim}{x}_D)| \leq B_i||\underset{\sim}{x}_D||_\kappa^\nu$.
      Hence, for i large enough,
      $\forall \underset{\sim}{x}_D \in \Pi_D, |\mathcal{U}(\underset{\sim}{x}_D)| \leq (B_i + \delta)||\underset{\sim}{x}_D||_\kappa^\nu$.
      Hence, ${}_\nu||\mathcal{U}|| = sup\{|\mathcal{U}(\underset{\sim}{x}_D)|/||\underset{\sim}{x}_D||_\kappa^\nu\} \leq (B_i + \delta) < \infty$
    * $\mathcal{U}$ continuous at $\underset{\sim}{x}_D \in \Pi_D$:
      need to show that for any $\xi > 0, \exists \delta > 0$ such that if $||\underset{\sim}{x}_D - \underset{\sim}{y}_D||_\kappa^\nu \leq \delta$, then $|\mathcal{U}(\underset{\sim}{x}_D) - \mathcal{U}(\underset{\sim}{y}_D)| \leq \xi$.

Given ξ and $\underset{\sim}{x}_D$, choose i big enough such that
${}_\nu||\mathcal{U}_i - \mathcal{U}|| \leq (\xi/4)/||\underset{\sim}{x}_D||^\nu_\kappa$,
and choose δ such that
if $||\underset{\sim}{x}_D - \underset{\sim}{y}_D||^\nu_\kappa \leq \delta$, then $|\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}_i(\underset{\sim}{y}_D)| < \xi/4$ and $(||\underset{\sim}{y}_D||^\nu_\kappa/||\underset{\sim}{x}_D||^\nu_\kappa) \leq 2$.
Hence, we have

$$\begin{aligned}|\mathcal{U}(\underset{\sim}{x}_D) - \mathcal{U}(\underset{\sim}{y}_D)| &\leq |\mathcal{U}(\underset{\sim}{x}_D) - \mathcal{U}_i(\underset{\sim}{x}_D)| + |\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}_i(\underset{\sim}{y}_D)| + |\mathcal{U}_i(\underset{\sim}{y}_D) - \mathcal{U}(\underset{\sim}{y}_D)| \\ &\leq {}_\nu||\mathcal{U}_i - \mathcal{U}||\ ||\underset{\sim}{x}_D||^\nu_\kappa + |\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}_i(\underset{\sim}{y}_D)| + {}_\nu||\mathcal{U}_i - \mathcal{U}||\ ||\underset{\sim}{y}_D||^\nu_\kappa \\ &\leq \xi/4 + \xi/4 + (\xi/4)(||\underset{\sim}{y}_D||^\nu_\kappa/||\underset{\sim}{x}_D||^\nu_\kappa) \leq \xi/4 + \xi/4 + \xi/2 = \xi.\end{aligned}$$

* $\mathcal{U}$ is h.o.d. ν:
  $|\mathcal{U}(\lambda \underset{\sim}{x}_D) - \lambda^\nu \mathcal{U}(\underset{\sim}{x}_D)| \leq |\mathcal{U}(\lambda \underset{\sim}{x}_D) - \mathcal{U}_i(\lambda \underset{\sim}{x}_D)| + \lambda^\nu |\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}(\underset{\sim}{x}_D)|$.
  For any ξ, choose i big enough so that
  $|\mathcal{U}(\lambda \underset{\sim}{x}_D) - \mathcal{U}_i(\lambda \underset{\sim}{x}_D)| \leq \xi/2$ and $|\mathcal{U}_i(\underset{\sim}{x}_D) - \mathcal{U}(\underset{\sim}{x}_D)| \leq (\xi/2)/\lambda^\nu$
  Hence, $|\mathcal{U}(\lambda \underset{\sim}{x}_D) - \lambda^\nu \mathcal{U}(\underset{\sim}{x}_D)| \leq \xi$. Since ξ was arbitrary, $\mathcal{U}(\lambda \underset{\sim}{x}_D) = \lambda^\nu \mathcal{U}(\underset{\sim}{x}_D)$.

Hence, $\mathcal{U} \in \mathbb{U}_D$.

Hence, $\mathbb{U}_D$ complete.

- $\mathcal{J} : \mathbb{U}_D \to \mathbb{U}_D$:

  - $(\mathcal{J}\mathcal{U})(.)$ is continuous because u_D and Ψ_D are continuous;
  - $(\mathcal{J}\mathcal{U})(.)$ is h.o.d. ν:

    $$\begin{aligned}(\mathcal{J}\mathcal{U})(\lambda \underset{\sim}{x}_{Dt}) = {} & u_D(\widetilde{C}^o(\lambda x_{Dt}, \lambda x_{Dt+1}), \lambda N^o_t) \\ & + \Psi_D[N^m_t, u_D(\widetilde{C}^m(\lambda x_{Dt}, x_{Dt+1}), N^m) + \mathcal{U}(\lambda \underset{\sim}{x}_{Dt+1})].\end{aligned}$$

    which can be done because the constraint in the definition of $\Pi(x_{D0}, \underset{\sim}{w}_t, \underset{\sim}{r}_t)$ is h.o.d. 1.

    Since C_D h.o.d. 1 in (x_{Dt}, x_{Dt+1}), we get

    $$\begin{aligned}(\mathcal{J}\mathcal{U})(\lambda \underset{\sim}{x}_{Dt}) = {} & u_D(\lambda \widetilde{C}^o(x_{Dt}, x_{Dt+1}), \lambda N^o_t) \\ & + \Psi_D[\lambda N^m_t, u_D(\lambda \widetilde{C}^m(x_{Dt}, x_{Dt+1}), \lambda N^m_t) + \mathcal{U}(\lambda x_{Dt+1})]\end{aligned}$$

    Since u_D and $\mathcal{U}$ are h.o.d. ν and Ψ_D is h.o.d. ν in the sense of Assumptions S.2.

    $$\begin{aligned}(\mathcal{J}\mathcal{U})(\lambda \underset{\sim}{x}_{Dt}) = {} & \lambda^\nu (u_D(\widetilde{C}^o(x_{Dt}, x_{Dt+1}), N^o_t) \\ & + \Psi_D[N^m_t, u_D(\widetilde{C}^m(x_{Dt}, x_{Dt+1}), N^m_t) + \mathcal{U}(\underset{\sim}{x}_{Dt+1})])\end{aligned}$$

    $$(\mathcal{J}\mathcal{U})(\lambda \underset{\sim}{x}_{Dt}) = \lambda^\nu (\mathcal{J}\mathcal{U})(\underset{\sim}{x}_{Dt}).$$

- ${}_\nu||\mathcal{J}\mathcal{U}|| < \infty$:

  By Assumption S.3, $\exists B$ such that

  $\forall \underset{\sim}{y}_t \in \Pi_D(y_t), |\mathcal{J}\mathcal{U}(\underset{\sim}{y}_{Dt})| \leq B\{(||y_{Dt}||_E)^\nu + (||y_{Dt+1}||_E)^\nu + |\mathcal{U}(\underset{\sim}{y}_{Dt+1})|\}$

  By definition of κ, $\kappa^\nu(||y_{Dt}||_E)^\nu \geq (||y_{Dt+1}||_E)^\nu$. Hence, we have

  $\forall \underset{\sim}{y}_t \in \Pi_D(y_t), |\mathcal{J}\mathcal{U}(\underset{\sim}{y}_{Dt})| \leq B\{(||y_{Dt}||_E)^\nu(1+\kappa^\nu) + |\mathcal{U}(\underset{\sim}{y}_{D_{t+1}})|\}$

  Let $\underset{\sim}{x}_{D0} = \underset{\sim}{y}_{Dt}/\kappa^t$, i.e. $y_{Dt+s} = \kappa^t x_{Ds}$. Then, since $\mathcal{J}\mathcal{U}(.)$ h.o.d. ν and Π h.o.d. 1, we have

  $\forall \underset{\sim}{x}_{D0} \in \Pi_D(x_0), (\kappa^\nu)^t|\mathcal{J}\mathcal{U}(\underset{\sim}{x}_{D0})| \leq B\{(\kappa^\nu)^t(||x_{D0}||_E)^\nu(1+\kappa^\nu) + |(\kappa^\nu)^{t+1}\mathcal{U}(\underset{\sim}{x}_{D0})|\}$.

  Divide both sides by $(\kappa^\nu)^t||\underset{\sim}{x}_{Dt}||_\kappa^\nu$ to get

  $\forall \underset{\sim}{x}_{D0} \in \Pi_D(x_0), \frac{|\mathcal{J}\mathcal{U}(\underset{\sim}{x}_{D0})|}{||\underset{\sim}{x}_{Dt}||_\kappa^\nu} \leq B\{\frac{(||x_{D0}||_E)^\nu}{||\underset{\sim}{x}_{Dt}||_\kappa^\nu}(1+\kappa^\nu) + (\kappa^\nu)\frac{|\mathcal{U}(\underset{\sim}{x}_{D0})}{||\underset{\sim}{x}_{Dt}||_\kappa^\nu}|\}$.

  Since, $\kappa > 0$ implies $||\underset{\sim}{x}_{Dt}||_\kappa^\nu = ||x_{D0}||_E$, we get

  $\forall \underset{\sim}{x}_{D0} \in \Pi_D(x_0), \frac{|\mathcal{J}\mathcal{U}(\underset{\sim}{x}_{D0})|}{||\underset{\sim}{x}_{Dt}||_\kappa^\nu} \leq B\{(||x_{D0}||_E)^{\nu-1}(1+\kappa^\nu) + (\kappa^\nu)\frac{|\mathcal{U}(\underset{\sim}{x}_{D0})}{||\underset{\sim}{x}_{D0}||_E}\}$.

  Since $\mathcal{U} \in \mathbb{U}_D$, $|\mathcal{U}(\underset{\sim}{x}_{D0})|/||x_{D0}||_E < \infty$.

  Hence, $\forall x_{D0} \in \mathbb{X}_D, \forall \underset{\sim}{x}_{D0} \in \Pi_D(x_{D0}), |\mathcal{J}\mathcal{U}(\underset{\sim}{x}_{D0})|/||\underset{\sim}{x}_{Dt}||_\kappa^\nu < \infty$ and

  ${}_\nu||\mathcal{J}\mathcal{U}|| = \sup\{(|\mathcal{J}\mathcal{U}(\underset{\sim}{x}_{D0})|/||\underset{\sim}{x}_{Dt}||_\kappa^\nu : x_{D0} \in \mathbb{X}_D\} < \infty$.

- $\mathcal{J}$ is a contraction of modulus ζ.

  Given $a > 0$ and $\mathcal{U} \in \mathbb{U}_D$, define $(\mathcal{U}+a)(x_D) \equiv \mathcal{U}(\underset{\sim}{x}_D) + a||x_D||_\kappa^\nu$. Note that $(\mathcal{U}+a)(.) \in \mathbb{U}_D$.

  Lemma T.1 *(Modified Blackwell) Let $M : \mathbb{U}_D \to \mathbb{U}_D$, let $\mathcal{U}, G \in \mathbb{U}_D$.*

  If (i) M is monotone, i.e. $\mathcal{U} \geq G \Rightarrow M\mathcal{U} \geq MG$ and

  if (ii) M discounts, i.e. $M(\mathcal{U}+a) \leq M\mathcal{U} + \theta a$,

  then M is a contraction of modulus θ.

  Proof. The proof is standard except for the definition of $(\mathcal{U}+a)(.)$. See Alvarez (1994) (Proposition 3). ■

  - $\mathcal{J}$ is monotone: follows immediately from Assumption S.2(b).
  - $\mathcal{J}$ discounts:

    $$\mathcal{J}(\mathcal{U}+a)(\underset{\sim}{x}_{Dt}) = u_D(\widetilde{C}^o(x_{Dt}, x_{Dt+1}), N^o) + \Psi_D[N_t^m, u_D(\widetilde{C}^m(x_{Dt}, x_{Dt+1}), N_t^m) + (\mathcal{U}+a)(x_{Dt+1})]$$

    By the definition of $(\mathcal{U}+a)(.)$,

$$\mathcal{J}(\mathcal{U}+a)(\underset{\sim}{x}_{Dt}) = u_D(\widetilde{C}^o(\underset{\sim}{x}_{Dt}, x_{Dt+1}), N_t^o) + \Psi_D[N_t^m, u_D(\widetilde{C}^m(x_{Dt}, x_{Dt+1}), N_t^m) + \mathcal{U}(\underset{\sim}{x}_{Dt}) + a||\underset{\sim}{x}_{Dt+1}||_\kappa^\nu]$$

Since, $\kappa > 0$ implies $||\underset{\sim}{x}_{Dt}||_\kappa^\nu = ||x_{D0}||_E$ and since Ψ_D discounts, we get

$$\mathcal{J}(\mathcal{U}+a)(\underset{\sim}{x}_{Dt}) \leq u_D(\widetilde{C}^o(\underset{\sim}{x}_{Dt}, x_{Dt+1}), N_t^o) + \Psi_D[N_t^m, u_D(\widetilde{C}^m(x_{Dt}, x_{Dt+1}), N_t^m) + \mathcal{U}(\underset{\sim}{x}_{Dt})] + a\zeta||\underset{\sim}{x}_{D0}||_E$$

$$\mathcal{J}(\mathcal{U}+a)(\underset{\sim}{x}_{Dt}) \leq (\mathcal{J}\mathcal{U})(\underset{\sim}{x}_{Dt}) + \zeta a||x_{D0}||_E = (\mathcal{J}\mathcal{U})(\underset{\sim}{x}_{Dt}) + \zeta a||x_{Dt}||_\alpha^\nu = (\mathcal{J}\mathcal{U} + \zeta a)(\underset{\sim}{x}_{Dt})$$

Hence, $\mathcal{J}$ discounts.

Hence, $\mathcal{J}$ is a contraction of modulus ζ.

Hence, $\mathcal{J}$ has a unique fixed point, $U_D^o = \mathcal{J}U_D^o$.

Step 2: Given U_D^o, for all $x_0 \in \mathbb{R}_+^2$, $\underset{\sim}{x}_0 \in \Pi(x_0, \underset{\sim}{w}_0, \underset{\sim}{r}_0)$, define

$$U^o(\underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t) = \frac{U_D^o(\underset{\sim}{x}_{Dt}; \underset{\sim}{w}_t, \underset{\sim}{r}_t)}{(N_t^o)^\nu}.$$

where $x_{Dt} = N_t^o(x_t, 1)$.

- *Claim i*: U^o solves equation (S.3).

  Proof of Claim i:

$$U^o(\underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t) = \frac{U_D^o(\underset{\sim}{x}_{Dt}; \underset{\sim}{w}_t, \underset{\sim}{r}_t)}{(N_0^o)^\nu} = \frac{\beta u_D(r_t S_t - \widetilde{B}(.)w_t, N_t^m)}{(N_t^o)^\nu} + \frac{\Psi_D(N_t^m, u(w_t(N_t^m + \widetilde{B}(.)) - S_{t+1} - \theta_t N_{t+1}^m) + U_D^o(\underset{\sim}{x}_{Dt+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1}))}{(N_t^o)^\nu}.$$

  By definition of U^o and u_D,

$$= \frac{\beta (N_t^o)^\nu u([r_t S_t - \widetilde{B}(.)w_t]/N_t^o)}{(N_t^o)^\nu} + \frac{\Psi_D(N_t^m, (N_t^m)^\nu u([w_t(N_t^m + \widetilde{B}(.)) - S_{t+1} - \theta_t N_{t+1}^m]/N_t^m) + (N_{t+1}^o)^\nu U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1}))}{(N_t^o)^\nu}.$$

Since $N^o_{t+1} = N^m_t = N^o_t \frac{N^m_t}{N^o_t} = N^o_t n_{t-1}$, Ψ_D h.o.d. ν and by definition of Ψ_D, S_t and B_t

$$
\begin{aligned}
&= (N^o_t)^\nu \Big[\frac{\beta u([r_t S_t - \widetilde{B}(.) w_t]/N^o_t)}{(N^o_t)^\nu} \\
&\quad + \frac{\Psi_D(n_{t-1}, (n_{t-1})^\nu \{u([w_t(N^m_t + \widetilde{B}(.)) - S_{t+1} - \theta_t N^m_{t+1}]/N^m_t) + U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1})\})}{(N^o_t)^\nu} \Big] \\
&= \beta u([r_t S_t - \widetilde{B}(.) w_t]/N^o_t) \\
&\quad + \Psi(n_{t-1}, u([w_t(N^m_t + \widetilde{B}(.)) - S_{t+1} - \theta_t N^m_{t+1}]/N^m_t) + U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1})) \\
&= \beta u(r_t s_t - w_t n_{t-1} b(.)) + \Psi(n_{t-1}, u(w_t(1 + b(.)) - s_{t+1} - \theta n_t) + U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1})).
\end{aligned}
$$

Hence, U^o solves equation (S.3).

- *Claim ii*: U^o is unique.

  Proof of Claim ii: Suppose there was another function, $\widetilde{U}^o$ that solves equation (S.3). Then $\widetilde{U}^o_D \equiv (N^o)^\nu \widetilde{U}^o$ would be different from U^o_D but solve equation (S.10): a contradiction since U^o_D is unique.

Step 3: Given U^o, for all $x_0 \in \mathbb{R}^2_+$, $\underset{\sim}{x}_0 \in \Pi(x_0, \underset{\sim}{w}_0, \underset{\sim}{r}_0)$ and for $b_t = \widetilde{b}(x_t, x_{t+1}; w_t, r_t)$, define

$$U^m(b_t, \underset{\sim}{x}_t; \underset{\sim}{w}_t, \underset{\sim}{r}_t) = u(w_t(1 + b_t) - s_{t+1} - \theta n_t) + U^o(\underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1}).$$

- *Claim iii*: U^m solves equation (4).

  Proof of Claim iii:

  Using equation (S.3),

$$
\begin{aligned}
&= u(w_t(1 + b_t) - s_{t+1} - \theta n_t) + \beta u(r_{t+1} s_{t+1} - w_{t+1} n_t b(.)) \\
&\quad + \Psi(n_t, u(w_{t+1}(1 + b(.)) - s_{t+2} - \theta n_{t+1}) + U^o(\underset{\sim}{x}_{t+2}; \underset{\sim}{w}_{t+2}, \underset{\sim}{r}_{t+2})).
\end{aligned}
$$

  Since $b_{t+1} = b(x_{t+1}, x_{t+2}; w_{t+1}, w_{t+2})$,

$$
\begin{aligned}
&= u(w_t(1 + b_t) - s_{t+1} - \theta n_t) + \beta u(r_{t+1} s_{t+1} - w_{t+1} n_t b_{t+1}) \\
&\quad + \Psi(n_t, u(w_{t+1}(1 + b_{t+1}) - s_{t+2} - \theta n_{t+1}) + U^o(\underset{\sim}{x}_{t+2}; \underset{\sim}{w}_{t+2}, \underset{\sim}{r}_{t+2})).
\end{aligned}
$$

By definition of U^m,

$$= u(w_t(1+b_t) - s_{t+1} - \theta n_t) + \beta u(r_{t+1}s_{t+1} - w_{t+1}n_t b_{t+1}) + \Psi(n_t, U^m(b_{t+1}, \underset{\sim}{x}_{t+1}; \underset{\sim}{w}_{t+1}, \underset{\sim}{r}_{t+1}).$$

Hence, U^m solves equation (4).

- *Claim iv*: U^m is unique.

  Proof of Claim iv: Suppose there was another function, $\widetilde{U}^m$ that solves equation (4). Then $\widetilde{U}^o_{t+1} \equiv \widetilde{U}^m_t - u(w_t(1+b_t) - s_{t+1} - \theta n_t)$ would be different from U^o but solve equation (S.3): a contradiction since U^o is unique.

■

T.2 Detailed proof of Proposition S.2

In this section we provide the details for the proof of Proposition S.2. For necessity of equations (S.22) and (S.23), we closely follow the arguments in Alvarez (1994), Prop. 8. For sufficiency of equations (S.22) and (S.23), we closely follow the proof of Theorem 4.15 in Stokey, Lucas, and Prescott (1989), p. 98.

Remark 2 *Note that Step 1 in Proposition S.1 can be used to show that U_P^o is uniquely defined through equation (S.19).*

Proposition:
If $M_P = F_N$, any sequence $\underset{\sim}{x}_{P0} \in \Pi_P(x_{P0})$ is optimal for the problem in Definition S.4 if and only if it satisfies the intra-temporal allocation condition (S.21), the Euler equations (S.22) and the transversality condition (S.23).

Proof. Necessity: Suppose an allocation attains the sup in Definition S.4, $U_P^{**}(x_{P0})$ where $U_P^{**} = U_P^*$ by Lemma S.3 and where $\widetilde{B}_P(.)$ is defined in equation (S.17). Then, since U_P^{o*} is strictly increasing, concave and differentiable by Lemma S.4, the following conditions hold necessarily:

$$B: \quad \beta\frac{\partial u_D(C^o, N^o)}{\partial C^o} = \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U_P^{*'})}{\partial U}\frac{\partial u_D(C^m, N^m)}{\partial C^m} + \lambda_B/M_P$$

$$K': \quad \frac{\partial u_D(C^m, N^m)}{\partial C^m} = \frac{\partial U_P^{*'}}{\partial K'}$$

$$N^{m'}: \quad \theta\frac{\partial u_D(C^m, N^m)}{\partial C^m} = \frac{\partial U_P^{*'}}{\partial N^{m'}}$$

where the first equation is equation (S.20) for $z = U_P^{*\prime}$.
The envelope conditions for the Pseudo-Planner are given by

$$
\begin{aligned}
K: \quad & \frac{\partial U_P^*}{\partial K} = \beta \frac{\partial u_D(C^o, N^o)}{\partial C^o} F_K \\
N^o: \quad & \frac{\partial U_P^*}{\partial N^o} = \beta \frac{\partial u_D(C^o, N^o)}{\partial N^o} \\
N^m: \quad & \frac{\partial U_P^*}{\partial N^m} = \frac{\partial u_D(C^o, N^o)}{\partial C^o}(F_N - M_P) + \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U_P^{*\prime})}{\partial N^m} \\
& \qquad + \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U_P^{*\prime})}{\partial U} \left[M_P \frac{\partial u_D(C^m, N^m)}{\partial C^m} + \frac{\partial u_D(C^m, N^m)}{\partial N^m} + \frac{\partial U_P^{*\prime}}{\partial N^{o\prime}} \right] \\
& \qquad - \left(\beta \frac{\partial u_D(C^o, N^o)}{\partial C^o} - \frac{\partial \Psi_D(N^m, u_D(C^m, N^m) + U_P^{*\prime})}{\partial U} \frac{\partial u_D(C^m, N^m)}{\partial C^m} \right) \underline{b} M_P
\end{aligned}
$$

Substituting out $\frac{\partial U_P^*}{\partial x}$ and setting $M_P = F_N$, we get the conditions in (S.21) and (S.22).

To show that the transversality condition must hold at an optimum, consider the following claim.

Claim 3 *For any feasible sequence, $\underset{\sim}{x}_{Pt}$, that follows x_{P0}, we have $\zeta^t U_P^o(\underset{\sim}{x}_{Pt}) \to 0$ as $t \to \infty$.*

Proof of Claim: Using Assumption S.2(e) (i.e. Assumption S.3), using $||x_{Pt}||_E \leq (\kappa^\nu)^t ||x_{P0}||_E$ (by Remark 1), the fact that $\kappa > 0$ implies $||\underset{\sim}{x}_{Pt}||_\kappa^\nu = ||x_{P0}||_E$, knowing that $_\nu||U_P^o|| < \infty$ (by Remark 2 and Proposition S.1), multiplying by ζ^t and using $\zeta\kappa^\nu < 1$, the result follows.

In particular, $\zeta^t U_P^{o**}(x_{Pt}) \to 0$ for any feasible x_{Pt} starting from x_{P0}. But $U_P^{o**}(x_{Pt}) = U_P^{o*}(x_{Pt})$. Since $U_P^{o*}(x_{Pt})$ is hod ν, we have:

$$
\nu U_P^{o*}(x_{Pt}) = \frac{\partial U_{Pt}^{o*}(x_{Pt})}{\partial K_t} K_t + \frac{\partial U_{Pt}^{o*}(x_{Pt})}{\partial N_t^m} N_t^m + \frac{\partial U_{Pt}^{o*}(x_{Pt})}{\partial N_t^o} N_t^o.
$$

Hence,

$$
\zeta^t \left[\frac{\partial U_{Pt}^{o*}(x_{Pt})}{\partial K_t} K_t + \frac{\partial U_{Pt}^{o*}(x_{Pt})}{\partial N_t^m} N_t^m + \frac{\partial U_{Pt}^{o*}(x_{Pt})}{\partial N_t^o} N_t^o \right] \to 0 \text{ as } t \to \infty.
$$

Using the envelope conditions as well as the Euler equations, we get the transversality condition in (S.23) as desired.

Sufficiency: Suppose $\underset{\sim}{x}_{P0}^*$ satisfies equations (S.22) and (S.23). Define

$$
D_t \equiv U_P^o(\underset{\sim}{x}_{Pt}^*) - U_P^o(\underset{\sim}{x}_{Pt})
$$

some $\underset{\sim}{x}_{Pt} \in \Pi_P(x_{Pt})$. All we need to show is that $D_0 \geq 0, \forall \underset{\sim}{x}_{P0} \in \Pi_P(x_{P0})$. By concavity of $u_D(o,.)$, Ψ_D and $u_D(m,.)$ we can write Again, by concavity of $u_D(m,.)$, we have

$$D_t \quad \geq \quad \Xi_t + \frac{\partial \Psi_D(t,^*)}{\partial U} D_{t+1}$$

where Ξ_t contains a bunch of utility derivatives evaluated at x^*_P multiplied by $(x^*_{Pt} - x_{Pt})$. Sequentially using this inequality by substituting for D_{t+1} from $t = 0$ (with the standard convention that $\prod_{s=0}^{-1} \frac{\partial \Psi_D(s,^*)}{\partial U} = 1$) gives:

$$D_0 \geq \lim_{T\to\infty} \Big\{ \sum_{t=0}^{T} \left(\prod_{s=0}^{t-1} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) \Xi_t + \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) D_{T+1} \Big\}$$

Since the initial condition is given, we have $x^*_{P0} = x_{P0}$. Hence, several terms disappear in $t = 0$. Rearranging terms over time and grouping by time period (as opposed to age), it is straightforward to see that most of Ξ_t cancels using equation (S.21). What remains is:

$$\begin{aligned}
D_0 \quad \geq \quad & -\lim_{T\to\infty} \Big\{ \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) * \Big[\frac{\partial u_D(C^{m*}_T, N^{m*}_T)}{\partial C^{m*}_T} ((K^*_{T+1} - K_{T+1}) + \theta(N^{m*}_{T+1} - N^m_{T+1})) \\
& \qquad + \beta \frac{\partial u_D(C^{o*}_{T+1}, N^{o*}_{T+1})}{\partial N^{o*}_{T+1}} (N^{o*}_{T+1} - N^o_{T+1}) \Big] \Big\} \\
& + \lim_{T\to\infty} \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) D_{T+1} \\
\geq \quad & -\lim_{T\to\infty} \Big\{ \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) * \Big[\frac{\partial u_D(C^{m*}_T, N^{m*}_T)}{\partial C^{m*}_T} (K^*_{T+1} + \theta N^{m*}_{T+1}) + \beta \frac{\partial u_D(C^{o*}_{T+1}, N^{o*}_{T+1})}{\partial N^{o*}_{T+1}} N^{o*}_{T+1} \Big] \Big\} \\
& + \lim_{T\to\infty} \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) D_{T+1}
\end{aligned}$$

where the last inequality follows from the monotonicity of Ψ_D and u_D.
Using the EE for K, we get:

$$\begin{aligned}
D_0 \quad \geq \quad & -\beta \lim_{T\to\infty} \Big\{ \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) * \Big[\frac{\partial u_D(C^{o*}_{T+1}, N^{o*}_{T+1})}{\partial C^{o*}_{T+1}} F_{KT+1} (K^*_{T+1} + \theta N^{m*}_{T+1}) + \frac{\partial u_D(C^{o*}_{T+1}, N^{o*}_{T+1})}{\partial N^{o*}_{T+1}} N^{o*}_{T+1} \Big] \Big\} \\
& + \lim_{T\to\infty} \left(\prod_{s=0}^{T} \frac{\partial \Psi_D(s,^*)}{\partial U} \right) D_{T+1}
\end{aligned}$$

This almost immediately leads to the result using Claims 3 and 4.

Claim 4 $\frac{\partial \Psi_D(N^m,U)}{\partial U} \leq \zeta < 1$

Proof of Claim: By Assumption S.2(c)., one can easily show (taking derivatives wrt λ and setting $\lambda = 1$) that:

$$\frac{\partial \Psi_D(N^m,U)}{\partial N^m} N^m + \nu \frac{\partial \Psi_D(N^m,U)}{\partial N^m} U = \nu \Psi_D(N,U). \tag{T.1}$$

By Assumption S.2(d), Ψ_D discounts at rate $\zeta < 1$. Using the homogeneity property in equation (T.1) and taking selective limits for $a \to 0$ in the discounting inequality, we get the desired result.

Using Claims 4 and then 3, we get that the second term converges to 0. Using Claim 4, it then follows from the transversality condition in equation (S.23) that the first term goes to 0 as well. Hence, $D_0 \geq 0$. ■

T.3 Algebra details on the equivalence between equations (S.21) to (S.23) and equations (8) to (11)

Note that

$$u_D(C,N) = N^\nu u(C/N)$$
$$\partial u_D/\partial C = N^{\nu-1}u'(c)$$
$$\partial u_D/\partial N = N^{\nu-1}(\nu u(c) - cu'(c))$$
$$\begin{aligned}\Psi_D(N^m, u_D(C^m,N^m) + U_P^{o*'}) &= \Psi(N^m, (u_D(C^m,N^m) + U_P^{o*'})/(N^m)^\nu) \\ &= \Psi(N^o n, (N^m)^\nu (u(c^m) + U^{o*'})/(N^m)^\nu) \\ &= (N^o)^\nu \Psi(n, u(c^m) + U^{o*'})\end{aligned}$$
$$\begin{aligned}\partial \Psi_D/\partial N^m &= \Psi_n(N^m, (u_D(C^m,N^m) + U_P^{o*'})/(N^m)^\nu) \\ &\quad - \Psi_U(N^m, (u_D(C^m,N^m) + U_P^{o*'})/(N^m)^\nu)\nu(N^m)^{-\nu-1}(u_D(C^m,N^m) + U_P^{o*'}) \\ &= (N^o)^{\nu-1}\Psi_n(n, u(c^m) + U^{o*'}) \\ &\quad -(N^o)^\nu \Psi_U(n, u(c^m) + U^{o*'})\nu(N^m)^{-1}(u(c^m) + U^{o*'}) \\ &= (N^o)^{\nu-1}(\Psi_n(n, u(c^m) + U^{o*'}) - \Psi_U(n, u(c^m) + U^{o*'})\nu n^{-1}(u(c^m) + U^{o*'}))\end{aligned}$$
$$\begin{aligned}\partial \Psi_D/\partial U &= (N^m)^{-\nu}\Psi_U(N^m, (u_D(C^m,N^m) + U_P^{o*'})/(N^m)^\nu) \\ &= (N^o)^\nu (N^m)^{-\nu}\Psi_U(n, u(c^m) + U^{o*'}) = n^{-\nu}\Psi_U(n, u(c^m) + U^{o*'})\end{aligned}$$
$$\lambda_B = (N_t^m)^{\nu-1} n_{t-1}^{-\nu}\lambda_b$$

Intra-temporal allocation condition:

Using this, equation (S.21) becomes:

$$\beta(N_t^o)^{\nu-1}u'(c_t^o) = (n_{t-1})^{-\nu}\Psi_U(N_t^m)^{\nu-1}u'(c_t^m) + (N_t^m)^{\nu-1}\lambda_{b,t}n_{t-1}^{-\nu}/F_{Nt}$$

$$\beta n_{t-1}^{1-\nu}u'(c_t^o) = (n_{t-1})^{-\nu}\Psi_U u'(c_t^m) + \lambda_{b,t}n_{t-1}^{-\nu}/F_{Nt}$$

$$\beta n_{t-1}u'(c_t^o) = \Psi_U(n_{t-1}, U_t)u'(c_t^m) + \lambda_{b,t}/F_{Nt}$$

Using the firm's optimality, this boils down to equation (8).

Euler Equations:

Equations (S.22) become:

$$\begin{aligned}
&(N_t^m)^{\nu-1}u'(c_t^m) = \beta(N_{t+1}^o)^{\nu-1}u'(c_{t+1}^o)F_{Kt+1} \\
&\theta(N_t^m)^{\nu-1}u'(c_t^m) = (N_t^m)^{\nu-1}(\Psi_n(n_t, U_{t+1}) - \Psi_U(n_t, U_{t+1})\nu n_t^{-1}U_{t+1}) \\
&\quad + n_t^{-\nu}\Psi_U(n_t, U_{t+1}) \\
&\quad * [F_{Nt+1}(N_{t+1}^m)^{\nu-1}u'(c_{t+1}^m) + (N_{t+1}^m)^{\nu-1}(\nu u(c_{t+1}^m) - c_{t+1}^m u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o))] \\
&\quad - \lambda_{b,t+1}(N_{t+1}^m)^{\nu-1}n_t^{-\nu}\underline{b}_{t+1}
\end{aligned}$$

Dividing both sides by $(N_t^m)^{\nu-1}$ in both equations gives:

$$\begin{aligned}
&u'(c_t^m) = \beta u'(c_{t+1}^o)F_{Kt+1} \\
&\theta u'(c_t^m) = \Psi_n(n_t, U_{t+1}) - n_t^{-1}\Psi_U(n_t, U_{t+1})\nu(u(c_{t+1}^m) + U_{t+2}^{o*}) \\
&\quad + n_t^{-1}\Psi_U(n_t, U_{t+1})[F_{Nt+1}u'(c_{t+1}^m) + (\nu u(c_{t+1}^m) - c_{t+1}^m u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o))] \\
&\quad - \lambda_{b,t+1}n_t^{-1}\underline{b}_{t+1}
\end{aligned}$$

The first equation boils down to equation 6. For the second, let's cancel some terms:

$$\begin{aligned}
&\theta u'(c_t^m) = \Psi_n(n_t, U_{t+1}) - n_t^{-1}\Psi_U(n_t, U_{t+1})\nu U_{t+2}^{o*} \\
&\quad + n_t^{-1}\Psi_U(n_t, U_{t+1})[(F_{Nt+1} - c_{t+1}^m)u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o))] \\
&\quad - \lambda_{b,t+1}n_t^{-1}\underline{b}_{t+1}
\end{aligned}$$

Using the budget constraint when young gives:

$$\theta u'(c_t^m) = \Psi_n(n_t, U_{t+1}) + n_t^{-1}\Psi_U(n_t, U_{t+1})$$
$$*[(s_{t+2} + \theta n_{t+1} - b_{t+1}w_{t+1})u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) - \nu U_{t+2}^{o*}]$$
$$-\lambda_{b,t+1} n_t^{-1} \underline{b}_{t+1}$$

Using the equation we derived from equation (S.21)

$$\theta u'(c_t^m) = \Psi_n(n_t, U_{t+1}) - n_t^{-1}(\beta n_t u'(c_{t+1}^o) - \lambda_{b,t+1}/F_{Nt+1})b_{t+1}w_{t+1}$$
$$+n_t^{-1}\Psi_U(n_t, U_{t+1})[(s_{t+2} + \theta n_{t+1})u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) - \nu U_{t+2}^{o*}]$$
$$-\lambda_{b,t+1} n_t^{-1} \underline{b}_{t+1}$$

Now, clearly, the terms in $\lambda_{b,t+1}$ cancel out because either the constraint is not binding and $\lambda_{b,t+1} = 0$ or it is binding and $b_{t+1} = \underline{b}_{t+1}$.

$$\theta u'(c_t^m) = \Psi_n(n_t, U_{t+1}) - n_t^{-1}\beta n_t u'(c_{t+1}^o) b_{t+1}w_{t+1}$$
$$+n_t^{-1}\Psi_U(n_t, U_{t+1})[(s_{t+2} + \theta n_{t+1})u'(c_{t+1}^m) + \beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) - \nu U_{t+2}^{o*}]$$

We know, from the envelope condition, that

(1) $\partial U_{Pt+2}^*/\partial N^o = \beta \partial u_D(o, t+2)/\partial N^o = (N_{t+2}^o)^{\nu-1}\beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o))$

and, from the homogeneity of U_P^*, that

(2) $\partial U_{Pt+2}^*/\partial N_{t+2}^o = \nu U_{Pt+2}^*/N_{t+2}^o - (\partial U_{Pt+2}^*/\partial K_{t+2})K_{t+2}/N_{t+2}^o - (\partial U_{Pt+2}^*/\partial N_{t+2}^m)N_{t+2}^m/N_{t+2}^o$

Or, using the FOCs for K and N,

(2) $\partial U_{Pt+2}^*/\partial N_{t+2}^o = \nu U_{Pt+2}^*/N_{t+2}^o - \left(\frac{\partial u_D(C_{t+1}^m, N_{t+1}^m)}{\partial C^m}\right) s_{t+2} - \left(\frac{\partial u_D(C_{t+1}^m, N_{t+1}^m)}{\partial C^m}\right)\theta n_{t+1}$

$= (N_{t+2}^o)^{\nu-1}[\nu U_{t+2}^{o*} - u'(c_{t+1}^m)(s_{t+2} + \theta n_{t+1})]$

Hence, $\beta(\nu u(c_{t+2}^o) - c_{t+2}^o u'(c_{t+2}^o)) = \nu U_{t+2}^{o*} - u'(c_{t+1}^m)(s_{t+2} + \theta n_{t+1})$.

That is, the second term on the RHS cancels out and we get equation (7).

Transversality condition:

Using the relationships in the beginning of the section and dividing by N_t^o, equation (S.23) becomes:

$$\zeta^t (N_t^o)^{\nu-1}[u'(c_t^o)F_{Kt}(K_t + \theta N_t^m) + (\nu u(c_t^o) - c_t^o u'(c_t^o))N_t^o] \underset{t\to\infty}{\to} 0$$

Dividing by $(N_t^o)^\nu$,

$$\zeta^t[u'(c_t^o)F_{Kt}(s_t+\theta n_{t-1})+\nu u(c_t^o)-c_t^o u'(c_t^o)] \underset{t\to\infty}{\to} 0$$

Using the budget constraint when old:

$$\zeta^t[u'((F_{Kt}k_t-b_tF_{Nt})n_{t-1})(F_{Kt}\theta-b_tF_{Nt})n_{t-1})+\nu u((F_{Kt}k_t-b_tF_{Nt})n_{t-1})] \underset{t\to\infty}{\to} 0$$

That is, we get equation (11).

T.4 Detailed proof that U_P^{o*} is differentiable (Lemma S.4)

T.4.1 $U_P^*(K_P, N_P^m, N_P^o; M_P)$ differentiable in $K_P > 0$

Define
$G(A) \equiv \max_{C^m,C^o,B}\{\beta U(C^o,N_P^o)+\Psi_D[N_P^m,U(C^m,N_P^m)+U_P^*(x_P^*(x_P;M_P);M_P')]\}$
subject to

$C^m \leq M_P(N_P^m+B)-\theta N_P^{m*'}(x_P;M_P)-K^{*'}(x_P;M_P),$
$C^o \leq F(A,N_P^m)-M_P(N_P^m+B),$
$B \geq N_P^m\underline{b}$

Note that:

- for $A=K_P>0$, $F(A)=U_P^*(K_P,N_P^m,N_P^o;M_P)$.
- $(K_P^{*'}(x_P;M_P),N_P^{m*'}(x_P;M_P)) \in \Gamma_P(A+h,N_P^m,N_P^o;M_P)$,
- $(K_P^{*'}(x_P;M_P),N_P^{m*'}(x_P;M_P)) \in \Gamma_P(A-h,N_P^m,N_P^o;M_P)$ for h small enough (since, at an optimum, $C^o>0$ because $\frac{\partial u_D(C^o,N^o)}{\partial C^o}_{|C^o=0}=\infty$),
- $G(A+\varepsilon) \leq U_P^*(K_P+\varepsilon,N_P^m,N_P^o;M_P), \forall\varepsilon\in[-h,h]$,
- since u_D is strictly concave and Ψ_D weakly concave, F is strictly concave,
- since u_D and Ψ_D are differentiable, F is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\frac{\partial U_P^{o*}(x_P;M_P)}{\partial K_P}=G'(A)=\beta\frac{\partial u_D(C^o,N_P^o)}{\partial C^o}>0 \tag{T.2}$$

T.4.2 $U_P^*(K_P, N_P^m, N_P^o; M_P)$ differentiable in $N_P^o > 0$

Define

$G(A) \equiv \max_{C^m, C^o, B} \{\beta u_D(C^o, A) + \Psi_D[N_P^m, U(C^m, N_P^m) + U_P^*(K_P^{*'}(x_P; M_P), N_P^{m*'}(x_P; M_P), N_P^m)]\}$

subject to

$$C^m \leq M_P(N_P^m + B) - \theta N_P^{m*'}(x_P; M_P) - K^{*'}(x_P; M_P),$$
$$C^o \leq F(K_P, N_P^m) - M_P(N_P^m + B),$$
$$B \geq N_P^m \underline{b}$$

Note that:

- for $A = N_P^o > 0$, $G(A) = U_P^*(K_P, N_P^m, N_P^o; M_P)$,
- since Γ_P is independent on N_P^o,

  $(K_P^{*'}(x_P; M_P), N_D^{m*'}(x_P; M_P)) \in \Gamma_P(K_P, N_P^m, A + h; M_P)$,

  $(K_P^{*'}(x_P; M_P), N_D^{m*'}(x_P; M_P)) \in \Gamma_P(K_P, N_P^m, A - h; M_P)$,
- $G(A + \varepsilon) \leq U_P^*(K_P, N_P^m, N_P^o + \varepsilon; M_P), \forall \varepsilon \in [-h, h]$,
- since u_D is strictly concave, G is strictly concave,
- since u_D is differentiable, G is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\frac{\partial U_P^{o*}(x_P; M_P)}{\partial N_P^o} = G'(A) = \beta \frac{\partial u_D(C^o, N_P^o)}{\partial N^o} > 0. \tag{T.3}$$

T.4.3 $U_P^*(K_P, N_P^m, N_P^o; M_P)$ differentiable in $N_P^m > 0$

Define

$G(A) \equiv \max_{C^m, C^o, B} \{\beta u_D(C^o, N_P^o) + \Psi_D[A/M_P, u_D(C^m, A/M_P) + U_P^*(K_P^{*'}(x_P; M_P), N_P^{m*'}(x_P; M_P), A/M_P)]\}$

subject to

$$C^m \leq A + M_P B - \theta N_P^{m*'}(x_P; M_P) - K^{*'}(x_P; M_P),$$
$$C^o \leq F(K_P, A/M_P) - (A + M_P B),$$
$$M_P B \geq A\underline{b}.$$

Case (i): $M_P B > A\underline{b}$.

Note that:

- for $A = M_P N_P^m > 0$, $G(A) = U_P^*(K_P, N_P^m, N_P^o; M_P)$.
- for h small enough, $M_P B > (A+h)\underline{b}$ and $(K_P^{*'}(x_P, M_P), N_P^{m*'}(x_P, M_P)) \in \Gamma_P(K_P, (A+h)/M_P, N_P^o; M_P)$, (since, at an optimum, $C^o > 0$ because $\frac{\partial u_D(C^o,N^o)}{\partial C^o}_{|C^o=0} = \infty$),
- for h small enough, $M_P B > (A+h)\underline{b}$ and $(K_P^{*'}(x_P, M_P), N_P^{m*'}(x_P, M_P)) \in \Gamma_P(K_P, (A+h)/M_P, N_P^o; M_P)$, (since, at an optimum, $C^o > 0$ because $\frac{\partial u_D(C^o,N^o)}{\partial C^o}_{|C^o=0} = \infty$),
- $G(A+\varepsilon) \leq U_P^*(K_P, N_P^m + \varepsilon/M_P, N_P^o; M_P), \forall \varepsilon \in [-h, h]$,
- since u_D is strictly concave and Ψ_D weakly concave, G is strictly concave,
- since u_D and Ψ_D are differentiable in N^m and U_P^* differentiable in N^o, G is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\frac{\partial U_P^{o*}(x_P; M_P)}{\partial N_P^m} = G'(A)(dA/dN_P^m) = G'(A)M_P \tag{T.4}$$
$$= \beta \frac{\partial u_D(C^o, N_P^o)}{\partial C^o}(F_N(K_P, N_P^m) - M_P)$$
$$+\frac{\partial \Psi_D}{\partial N^m} + \frac{\partial \Psi_D}{\partial U}[M_P \frac{\partial u_D(C^m, N_P^m)}{\partial C^o} + \frac{\partial u_D(C^m, N_P^m)}{\partial N^m} + \beta \frac{\partial u_D(C^{o'}, N_P^{o'})}{\partial N^o}].$$

Or, since $F_N(K_P, N_P^m) = M_P$

$$\frac{\partial U_P^{o*}(x_P; M_P)}{\partial N_P^m} = G'(A)(dA/dN_P^m) = G'(A)M_P \tag{T.5}$$
$$= \frac{\partial \Psi_D}{\partial N^m} + \frac{\partial \Psi_D}{\partial U}[M_P \frac{\partial u_D(C^m, N_P^m)}{\partial C^o} + \frac{\partial u_D(C^m, N_P^m)}{\partial N^m} + \beta \frac{\partial u_D(C^{o'}, N_P^{o'})}{\partial N^o}] > 0.$$

Case (ii): $M_P B = A\underline{b}$ and $\lambda_B > 0$.

Note that:

- for $A = M_P N_P^m > 0$, $F(A) = U_D^*(K_D, N_D^m, N_D^o; \widehat{x})$.
- for h small enough, $M_P B = (A+h)\underline{b}$, $\lambda_B > 0$ and $(K_P^*(x_P, \widehat{x}), N_P^{m*}(x_P; M_P)) \in \Gamma_P(K_P, (A+h)/M_P), N_P^o; M_P)$, (since, at an optimum, $C^o > 0$ because $\frac{\partial u_D(C^o,N^o)}{\partial C^o}_{|C^o=0} = \infty$),

- for h small enough, $(K_D, (A-h)/w(\widehat{x}), N_D^o \in A_{CO}$ (since A_{CO} open) and $(K_D^*(x_D, \widehat{x}), N_D^{m*}(x_D, \widehat{x})) \in \Gamma_D(K_D, (A-h)/w(\widehat{x}), N_D^o; \widehat{x})$, (since, at an optimum, $C^m > 0$ because $\frac{\partial u_D(C^m,N^m)}{\partial C^m}_{|C^m=0} = \infty$),
- $G(A+\varepsilon) \leq U_P^*(K_P, N_P^m + \varepsilon/M_P, N_P^o; M_P), \forall \varepsilon \in [-h, h]$,
- since u_D is strictly concave and Ψ_D weakly concave, G is strictly concave,
- since u_D and Ψ_D are differentiable in N^m and U_P^* differentiable in N^o, G is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\begin{aligned}\frac{\partial U_P^{o*}(x_P; M_P)}{\partial N_P^m} &= G'(A)(dA/dN_P^m) = G'(A)M_P \\ &= \beta\frac{\partial u_D(C^o, N_P^o)}{\partial C^o}(F_N(K_P, N_P^m) - M_P(1+\underline{b}) + \frac{\partial \Psi_D}{\partial N^m} \\ &\quad + \frac{\partial \Psi_D}{\partial U}[M_P\frac{\partial u_D(C^m, N_P^m)}{\partial C^m}(1+\underline{b}) + \frac{\partial u_D(C^m, N_P^m)}{\partial N^m} + \beta\frac{\partial u_D(C^{o'}, N_P^{o'})}{\partial N^o}]\end{aligned} \tag{T.6}$$

Or, since $M_P = F_N(K_P, N_P^m)$

$$\begin{aligned}\frac{\partial U_P^{o*}(x_P; M_P)}{\partial N_P^m} &= \left[\frac{\partial \Psi_D}{\partial U}\frac{\partial u_D(C^m, N_P^m)}{\partial C^m} - \beta\frac{\partial u_D(C^o, N_P^o)}{\partial C^o}\right] M_P\underline{b} \\ &+ \frac{\partial \Psi_D}{\partial N^m} + \frac{\partial \Psi_D}{\partial U}[M_P\frac{\partial u_D(C^m, N_P^m)}{\partial C^m} + \frac{\partial u_D(C^m, N_P^m)}{\partial N^m} + \beta\frac{\partial u_D(C^{o'}, N_P^{o'})}{\partial N^o}] > 0\end{aligned} \tag{T.7}$$

[Note that the first term is $-\lambda_B\underline{b}$ and the rest is the same as in case (i).]

Case (iii): $M_PB = A\underline{b}$ and $\lambda_B = 0$.

Note that:

- for $A = M_PN_P^m > 0$, $G(A) = U_P^*(K_P, N_P^m, N_P^o; M_P)$.
- for and $h > 0$, $M_PB = (A+h)\underline{b}$ and $\lambda_B > 0$, i.e. we are in case (ii),
- for and $h < 0$, $M_PB > (A-h)\underline{b}$, i.e. we are in case (i),
- $G(A+\varepsilon) \leq U_P^*(K_P, N_P^m + \varepsilon/M_P, N_P^o; M_P), \forall \varepsilon \in [-h, h]$,

Hence, we need to show that

$\lim_{h>0, h\to 0} \frac{\partial U_P^{o*}(x_P;M_P)}{\partial N_P^m}_{|ii} = \lim_{h<0, h\to 0} \frac{\partial U_P^{o*}(x_P;M_P)}{\partial N_P^m}_{|i}$.

Note that $\lim_{h>0,h\to 0} \frac{\partial U_P^{o*}(x_P;M_P)}{\partial N_P^m}_{|ii} = -\lim_{h>0,h\to 0} \lambda_B \underline{b} + \frac{\partial U_P^{o*}(x_P;M_P)}{\partial N_P^m}_{|i}$.

Since, $\lambda_B \to 0$ as $h \to 0$, we have $\frac{\partial U_P^{o*}(x_P;M_P)}{\partial N_P^m}_{|iii} = \frac{\partial U_P^{o*}(x_P;M_P)}{\partial N_P^m}_{|i}$.

T.5 Detailed proof of Proposition S.3

Proposition:

U_P^* is differentiable with respect to the parameter $\underline{b}$ and the Pseudo Planner's policy functions are continuous in $\underline{b}$. Therefore, household policy functions and equilibrium prices are continuous in $\underline{b}$.

Proof.

Let's add $\underline{b}$ as a state variable with law of motion $\underline{b}' = \underline{b}$.

We want to show that $U_P^*(x_P;\underline{b}, M_P)$ differentiable in $\underline{b}$.

To do so, define

$F(A) \equiv \max_{C^m,C^o,B} \{\beta U(C^o, N_P^o) + \Psi_D[N_P^m, U(C^m, N_P^m) + U_P^*(x_P^*(x_P, \underline{b}))]\}$

subject to

$C^m \leq M_P(N_P^m + B) - \theta N^{m*}(x_P) - K^*(x_P),$

$C^o + M_P(N_P^m + B) \leq F(K_P, N_P^m),$

$M_P B \geq A$,

Define $\underline{b}^* \in [-1, \bar{b}]$ such that, given (K_P, N_P^m, N_P^o),

$$\beta \frac{\partial u_D(o, x_P, \underline{b}^*)}{\partial C^o} = \frac{\partial \Psi_D(N_P^m, u_D(m, x_P, \underline{b}^*)}{\partial U} + U_P^{o*}(x_P^*(x_P, \underline{b}^*))) \frac{\partial u_D(m, x_P, \underline{b}^*)}{\partial C^m}$$

and define $B^*(x_P, F_N(x_P)) = N_P^m \underline{b}^*$.

That is, the constraint is just binding. $\underline{b}^*$ exists if $\partial\Psi_D/\partial U > 0$. If not, only case ii is relevant.

i. Case 1: $\underline{b} < \underline{b}^*$.

   Note that:

   - for $A = M_P N_P^m \underline{b}$, $F(A) = U_P^*(x_D, \underline{b}; M_P)$.
   - for h small enough, x_P s.t. $\underline{b} < \underline{b}^*$ still holds for $(A \pm h)$
   - $(K_P^*(x_P, \underline{b}, M_P), N_P^{m*}(x_P, \underline{b}, M_P))$ feasible for $(A \pm h)$
     (since intertemporal constraint is independent on ($\underline{b}$.))
   - $F(A + \varepsilon) \leq U_P^*(x_P, (A + \varepsilon)/(M_P N_P^m); M_P), \forall \varepsilon \in [-h, h]$

- Since u_D is strictly concave and Ψ_D weakly concave, F is strictly concave.
- Since u_D and Ψ_D are differentiable in C, F is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989) , Theorem 4.10, p.84) to get that

$$\frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|case1} = F'(A)(dA/d\underline{b}) = 0$$

ii. Case 2: $\underline{b} > \underline{b}^*$.

Note that:

- for $A = M_P N_P^m \underline{b}$, $F(A) = U_P^*(x_D, \underline{b}; M_P)$.
- for h small enough, x_P s.t. $\underline{b} > \underline{b}^*$ still holds for $(A \pm h)$
- $(K_P^*(x_P, \underline{b}, M_P), N_P^{m*}(x_P, \underline{b}, M_P))$ feasible for $(A \pm h)$
  (since intertemporal constraint is independent on ($\underline{b}$.))
- $F(A + \varepsilon) \leq U_P^*(x_P, (A + \varepsilon)/(M_P N_P^m); M_P), \forall \varepsilon \in [-h, h]$
- Since u_D is strictly concave and Ψ_D weakly concave, F is strictly concave.
- Since u_D and Ψ_D are differentiable in C, F is differentiable.

Hence, we can apply the Benveniste-Scheinkman Theorem (Stokey, Lucas, and Prescott (1989), Theorem 4.10, p.84) to get that

$$\begin{aligned}
&\frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|case2} = F'(A)(dA/d\underline{b}) = F'(A) M_P N_P^m \qquad \text{(T.8)}\\
&= -\beta \frac{\partial u_D(C^o(x_P, x_P^*(x_P)), N_P^o)}{\partial C^o} M_P N_P^m \\
&+ \frac{\partial \Psi_D}{\partial U} * \frac{\partial u_D(C^m(x_P, x_P^*(x_P)), N_P^o)}{\partial C^m} M_P N_P^m \\
&= M_P N_P^m \lambda_B
\end{aligned}$$

iii. Case 3: $\underline{b} = \underline{b}^*$, i.e. $x_P \in A_{UC_P}$

Note that:

- for $A = M_P N_P^m \underline{b}$, $F(A) = U_P^*(x_D, \underline{b}; M_P)$.
- for and $h > 0$, we are back to case 1 for $(A + h)$ and

- for and $h < 0$, we are back to case 2 for $(A + h)$.
- $F(A + \varepsilon) \leq U_P^*(x_P, (A + \varepsilon)/(M_P N_P^m); M_P), \forall \varepsilon \in [-h, h]$

Hence, we need to show that

$$\lim_{h>0,h\to 0} \frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|case1} = \lim_{h<0,h\to 0} \frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|case2}$$

Since, $\lambda_B(x_P, x_P^*(x_P), M_P) \to 0$ as $h \to 0$, we have $\frac{\partial U_P^{o*}(x_P, \underline{b}, M_P)}{\partial \underline{b}}_{|case3} = 0$

Hence, U_P^* is differentiable in $\underline{b}$. Thus U_P^* is continuous in $\underline{b}$ and by the Theorem of the Maximum, the optimal policy functions, x_P^* are continuous in $\underline{b}$. Therefore, using the equivalence in Proposition S.5 equilibrium prices, $w(\widehat{x}'(\underline{b})) = F_N(x_P^*(\underline{b}))$ and $r(\widehat{x}'(\underline{b})) = F_K(x_P^*(\underline{b}))$, are continuous in $\underline{b}$. ■